9

Waveguide, Cavity and Microstrip Antenna

Scientists often have a naive faith that if only they could discover enough facts about a problem, these facts would somehow arrange themselves in a compelling and true solution.

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9.1 Introduction

High radiation losses are a problem for the conventional open wire lines at high frequency. So we usually use low-loss microwave transmission lines like coaxial cables, rectangular waveguide, circular waveguide, etc. We will investigate these transmission lines first and then move onto cavity and microstrip antennas. Before that let us try to develop a strong theoretical background which will be required for analysis of these transmission lines and transmission line based devices.

Note that for particular conductor geometry for a microwave transmission line, only certain patterns of electric and magnetic fields (also known as modes) can exist as propagating waves. These modes must be solutions to the governing differential equation (wave equation) while satisfying the appropriate boundary conditions for the fields. Coaxial cables have two conductors and we can define a unique current and voltage and characteristic impedance along the line using circuit equations. Whereas, waveguide typically has one enclosed conductor and we cannot define a unique voltage and current along the waveguide instead we must use fields to describe their operation.

The propagating modes along the transmission line or waveguide may be classified according to which field components are present or not present in the wave. The components of fields in the direction of wave propagation are defined as *longitudinal* components while those perpendicular to the direction of propagation are defined as *transverse* components.

For time-harmonic fields ($e^{j\omega t}$ time dependence), assuming wave propagation along the z-axis, the electric and magnetic fields can be written as

$$\vec{E}(x,y,z)=\left[\vec{E}_t(x,y)+\hat{z}E_z\right]e^{-j\beta z};\overline{H}(x,y,z)=\left[\vec{H}_t(x,y)+\hat{z}H_z\right]e^{-j\beta z} \quad (9.1)$$

where the first terms $\vec{E}_t(x,y)$ and $\vec{H}_t(x,y)$ represent the transverse components and second terms E_z and H_z represent the longitudinal components of the electric and magnetic fields respectively.

9.1.1 Transverse electromagnetic (TEM) modes

The electric and magnetic fields are transverse to the direction of wave propagation with no longitudinal components [$E_z = H_z = 0$]. TEM modes cannot exist on single conductor guiding structures. TEM modes are sometimes called *transmission line modes* since they are the dominant modes on transmission lines. *Quasi-TEM modes* – modes, which approximate true TEM modes when the frequency is sufficiently small.

$$\lim_{f\to 0} E_z = \lim_{f\to 0} H_z = 0 \quad (9.2)$$

9.1.2 Transverse electric (TE) modes

The electric field is transverse to the direction of propagation (no longitudinal electric field component) while the magnetic field has both transverse and longitudinal components [$E_z = 0$, $H_z \neq 0$].

9.1.3 Transverse magnetic (TM) modes

The magnetic field is transverse to the direction of propagation (no longitudinal magnetic field component) while the electric field has both transverse and longitudinal components [$H_z = 0$, $E_z \neq 0$].

TE and TM modes are commonly referred to as *waveguide modes* since they are the only modes, which can exist, in an enclosed guiding structure. TE and TM modes are characterized by a *cutoff frequency* below which they do not propagate. TE and TM modes can exist on transmission lines but are generally undesirable (higher order modes). Transmission lines are typically operated at frequencies below the cutoff frequencies of TE and TM modes so that only the TEM mode exists.

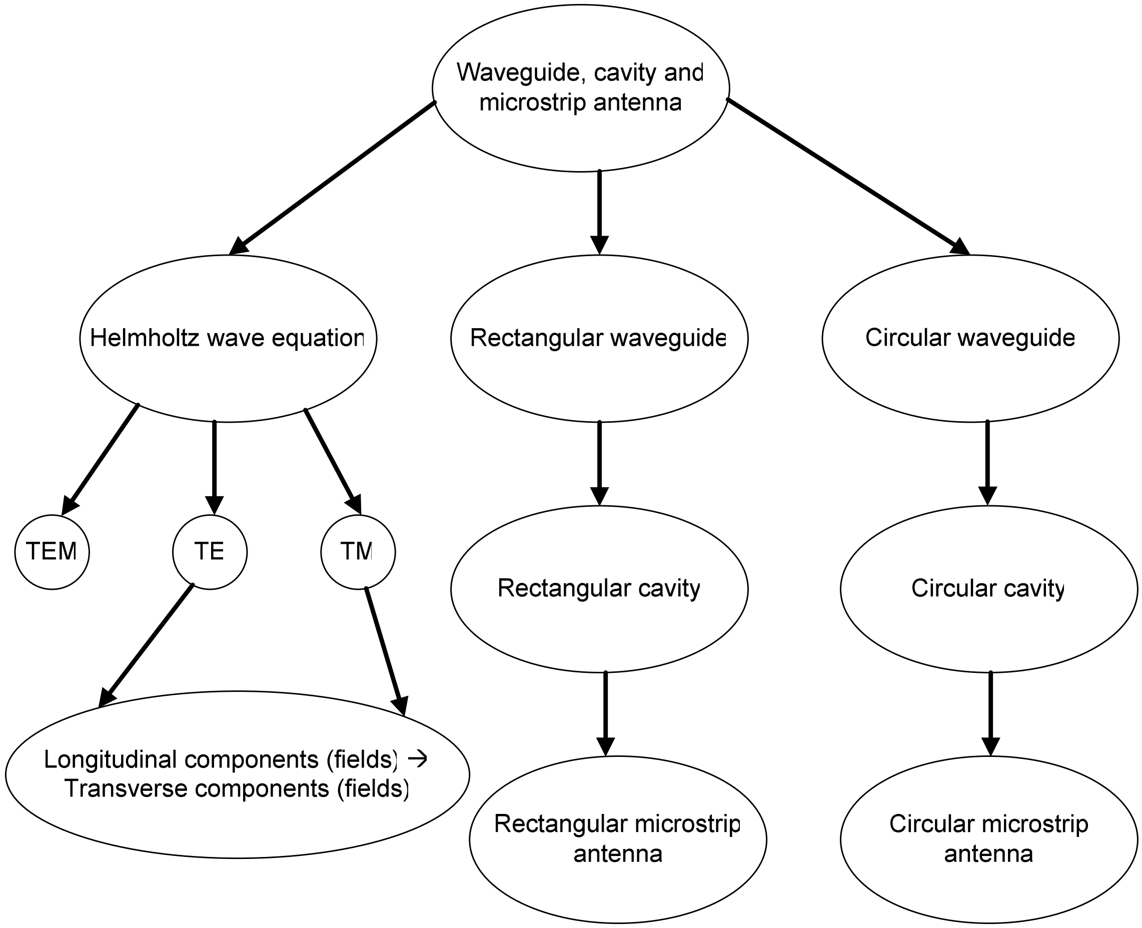


Fig. 9.1 Waveguide, cavity and microstrip antenna

9.1.4 Helmholtz wave equations

We may write general solutions to the fields associated with the waves that propagate on a guiding structure using Maxwell's equations. We assume the following about the guiding structure:

(1) it is infinitely long, oriented along the z-axis, and uniform along its length.

(2) it is constructed from ideal materials (conductors are Perfect Electric Conductor (PEC) and insulators are lossless).

(3) fields are time-harmonic.

The fields of the guiding structure must satisfy the source free Maxwell's equations. We can write down the two source-free Maxwell's equations in phasor form as follows:

$$\nabla\times\vec{E} = -j\omega\mu\vec{H} \tag{9.3a}$$

$$\nabla\times\vec{H} = j\omega\varepsilon\vec{E} \tag{9.3b}$$

Taking the curl of the first equation in (9.3a), expand it using the well known the vector identity, we have studied in chapter 1, and using the equation (9.3b), we have,

$$\nabla\times\nabla\times\vec{E} = \nabla\bullet(\nabla\bullet\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E} = -j\omega\mu\nabla\times\vec{H} = \omega^2\mu\varepsilon\vec{E} \tag{9.4}$$

Note that in the above equation we have also used the condition that $\nabla\bullet\vec{E} = 0$ in a source-free region. Now the equation (9.4) can be expressed as

$$\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = \nabla^2\vec{E} + k^2\vec{E} = 0 \tag{9.5}$$

which is the Helmholtz wave equation and $k = \omega\sqrt{\mu\varepsilon}$.

For time-harmonic fields ($e^{j\omega t}$ time dependence) with wave propagation along the z-axis, the electric and magnetic fields can be written as

$$\vec{E}(x, y, z) = \left[\vec{e}_t(x, y) + e_z\hat{z}\right]e^{-j\beta z}$$

$$\vec{H}(x, y, z) = \left[\vec{h}_t(x, y) + h_z\hat{z}\right]e^{-j\beta z}$$

where $\vec{e}_t(x, y)$ and $\vec{h}_t(x, y)$ represent the transverse electric and magnetic field components and e_z and h_z represent the longitudinal electric and magnetic field components. This is the notation we will be following.

For TEM waves (E_z = H_z =0), since $\beta = k = \omega\sqrt{\mu\varepsilon}$, we can further simplify the Helmholtz wave equation as below.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)E_x=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\beta^2+k^2\right)e_x=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)e_x=0 \quad (9.6)$$

Similarly, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)e_y=0$.

Therefore, $\nabla_t^2\vec{e}_t(x,y)=0$ (9.7)

which means that the transverse components of electric field satisfies the Laplace's equation. Similarly, it can be shown that the transverse components of the magnetic field ($\nabla_t^2\vec{h}_t(x,y)=0$) also satisfies the Laplace's equation.

Since, $\nabla_t\times\vec{e}_t(x,y)=-j\omega\mu h_z\hat{z}=0$, we can define $\vec{e}_t=-\nabla_t\phi(x,y)$. Note that $\nabla_t\times\vec{e}_t(x,y)$ will be directed along the z axis only since it involves unit vectors like $\hat{x}\times\hat{y},\quad \hat{y}\times\hat{x},\quad \hat{x}\times\hat{x}\quad and\quad \hat{y}\times\hat{y}$ only. For source free region,

$$\nabla\bullet\vec{D}=\varepsilon\nabla\bullet\vec{e}=\varepsilon\nabla_t\bullet\vec{e}_t=0\Rightarrow\nabla_t^2\phi(x,y)=0. \quad (9.8)$$

So the transmission lines which support TEM waves like coaxial cables can be analyzed easily from the Laplace's equation like in electrostatics. All we have to do is to solve the Laplace's equation to find its solutions for the TEM waves.

For TE waves (E_z=0), we can further simplify the Helmholtz wave equation as below.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)H_z=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\beta^2+k^2\right)h_z=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)h_z=0 \quad (9.9)$$

where $k_c^2=k^2-\beta^2$ gives us the cut-off frequency for various TE modes or waves.

For TM waves (H_z=0), we can further simplify the Helmholtz wave equation as below.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)E_z=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\beta^2+k^2\right)e_z=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)e_z=0 \qquad (9.10)$$

where $k_c^2=k^2-\beta^2$ gives us the cut-off frequency for various TM modes or waves.

9.1.5 TE/TM fields

You may be wondering why is that for both TE and TM waves we have simplified the wave equation for longitudinal components only. This is because once we get the longitudinal components of electric or magnetic fields, we can obtain the other components of the electric and magnetic fields. Let us do this analysis first in Cartesian coordinate system (suitable for rectangular waveguide) and then in Cylindrical coordinate system (suitable for circular waveguide).

Cartesian Coordinate Systems (suitable for rectangular waveguides):

Let us first write down the source-free Maxwell's curl equation

$$\nabla\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=\hat{i}\left(\frac{\partial}{\partial y}E_z-\frac{\partial}{\partial z}E_y\right)+\hat{j}\left(\frac{\partial}{\partial z}E_x-\frac{\partial}{\partial x}E_z\right)+\hat{k}\left(\frac{\partial}{\partial x}E_y-\frac{\partial}{\partial y}E_x\right)$$

$$=-j\omega\mu\left(H_x\hat{i}+H_y\hat{j}+H_z\hat{k}\right) \qquad (9.11)$$

Equating the x-, y- and z- components, we get,

$$\frac{\partial}{\partial y}E_z-\frac{\partial}{\partial z}E_y=-j\omega\mu H_x \qquad (9.12a)$$

$$\frac{\partial}{\partial z}E_x-\frac{\partial}{\partial x}E_z=-j\omega\mu H_y \qquad (9.12b)$$

$$\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x = -j\omega\mu H_z \tag{9.12c}$$

Similarly, from the second source-free Maxwell's curl equations, we have,

$$\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y = j\omega\varepsilon E_x \tag{9.12d}$$

$$\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y \tag{9.12e}$$

$$\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x = j\omega\varepsilon H_z \tag{9.12f}$$

Putting the expression of E_y from equation 9.12e to 9.12a and noting that $\frac{\partial}{\partial z} \equiv -j\beta$ for a wave propagating along z-axis, we obtain,

$$\frac{\partial}{\partial y}E_z + j\beta\left(\frac{-j\beta H_x - \frac{\partial}{\partial x}H_z}{j\omega\varepsilon}\right) = -j\omega\mu H_x$$

$$j\omega\varepsilon\frac{\partial}{\partial y}E_z + \beta^2 H_x - j\beta\frac{\partial}{\partial x}H_z = \omega^2\mu\varepsilon H_x$$

$$j\omega\varepsilon\frac{\partial}{\partial y}E_z - j\beta\frac{\partial}{\partial x}H_z = \left(k^2 - \beta^2\right)H_x = k_c^2 H_x$$

$$H_x = \frac{j\left(\omega\varepsilon\frac{\partial}{\partial y}E_z - \beta\frac{\partial}{\partial x}H_z\right)}{k_c^2} \tag{9.13a}$$

Similarly, we can obtain the other components of electric and magnetic fields from the longitudinal components and those final equations are listed below.

$$H_y = -\frac{j\left(\omega\varepsilon\frac{\partial}{\partial x}E_z + \beta\frac{\partial}{\partial y}H_z\right)}{k_c^2} \tag{9.13b}$$

$$E_x = -\frac{j\left(\beta \frac{\partial}{\partial x} E_z + \omega\mu \frac{\partial}{\partial y} H_z\right)}{k_c^2} \tag{9.13c}$$

$$E_y = \frac{j\left(-\beta \frac{\partial}{\partial y} E_z + \omega\mu \frac{\partial}{\partial x} H_z\right)}{k_c^2} \tag{9.13d}$$

How to remember the above formulae?

The above four equations could be rewritten as

$$E_x = \frac{1}{k_c^2}\left\{-\frac{\partial(f_1)}{\partial x} - \frac{\partial(f_2)}{\partial y}\right\}$$

$$E_y = \frac{1}{k_c^2}\left\{-\frac{\partial(f_1)}{\partial y} + \frac{\partial(f_2)}{\partial x}\right\}$$

$$H_x = \frac{1}{k_c^2}\left\{\frac{\partial(f_3)}{\partial y} - \frac{\partial(f_4)}{\partial x}\right\}$$

$$H_y = \frac{1}{k_c^2}\left\{-\frac{\partial(f_3)}{\partial x} - \frac{\partial(f_4)}{\partial y}\right\}$$

where $f_1 = j\beta E_z, f_2 = j\omega\mu H_z, f_3 = j\omega\varepsilon E_z, f_4 = j\beta H_z$

Cylindrical Coordinate Systems (suitable for circular waveguide):

In cylindrical coordinate system, the first source-free Maxwell's curl equation can be written as

$$\nabla \times \vec{E} = \frac{1}{s_1 s_2 s_3}\begin{vmatrix} s_1\hat{\rho} & s_2\hat{\phi} & s_3\hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ s_1 E_\rho & s_2 E_\phi & s_3 E_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{vmatrix}$$

Note that s_1, s_2 and s_3 are the scale factors of cylindrical coordinate system.

$$= \hat{\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}E_\phi\right) + \hat{\phi}\left(\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z\right) + \hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho\right)$$

$$= -j\omega\mu\left(H_\rho\hat{\rho} + H_\phi\hat{\phi} + H_z\hat{z}\right) \quad (9.14)$$

Equating the ρ-, ϕ- and z- components, we get,

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}E_\phi = -j\omega\mu H_\rho \quad (9.15a)$$

$$\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial\rho}E_z = -j\omega\mu H_\phi \quad (9.15b)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho = -j\omega\mu H_z \quad (9.15c)$$

This can be further simplified as

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z + j\beta E_\phi = -j\omega\mu H_\rho \quad (9.15d)$$

$$-j\beta E_\rho - \frac{\partial}{\partial\rho}E_z = -j\omega\mu H_\phi \quad (9.15e)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho = -j\omega\mu H_z \quad (9.15f)$$

Similarly from the second source-free Maxwell's curl equations, we have,

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}H_z + j\beta H_\phi = j\omega\varepsilon E_\rho \quad (9.15g)$$

$$-j\beta H_\rho - \frac{\partial}{\partial\rho}H_z = j\omega\varepsilon E_\phi \quad (9.15h)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho H_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}H_\rho = j\omega\varepsilon E_z \quad (9.15i)$$

Like the previous case, from equation 9.15d and 9.15h, we obtain,

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z + j\beta\left(\frac{-j\beta H_\rho - \frac{\partial}{\partial\rho}H_z}{j\omega\varepsilon}\right) = -j\omega\mu H_\rho$$

$$j\omega\varepsilon\frac{\partial}{\partial y}E_z - j\beta\frac{\partial}{\partial x}H_z = \left(k^2-\beta^2\right)H_x = k_c^2 H_x$$

$$H_\rho = \frac{j\left(\frac{\omega\varepsilon}{\rho}\frac{\partial}{\partial\phi}E_z - \beta\frac{\partial}{\partial\rho}H_z\right)}{k_c^2} \tag{9.16a}$$

Similarly, we can obtain the other components of electric and magnetic fields from the longitudinal components and those final equations are listed below.

$$H_\phi = -\frac{j\left(\omega\varepsilon\frac{\partial}{\partial\rho}E_z + \frac{\beta}{\rho}\frac{\partial}{\partial\phi}H_z\right)}{k_c^2} \tag{9.16b}$$

$$E_\rho = -\frac{j\left(\beta\frac{\partial}{\partial\rho}E_z + \frac{\omega\mu}{\rho}\frac{\partial}{\partial\phi}H_z\right)}{k_c^2} \tag{9.16c}$$

$$E_\phi = \frac{j\left(-\frac{\beta}{\rho}\frac{\partial}{\partial\phi}E_z + \omega\mu\frac{\partial}{\partial\rho}H_z\right)}{k_c^2} \tag{9.16d}$$

How to remember the above formulae?

The above four equations could be rewritten as

$$E_\rho = \frac{1}{k_c^2}\left\{-\frac{\partial(f_1)}{\partial\rho} - \frac{1}{\rho}\frac{\partial(f_2)}{\partial\phi}\right\}$$

$$E_\phi = \frac{1}{k_c^2}\left\{-\frac{1}{\rho}\frac{\partial(f_1)}{\partial\phi} + \frac{\partial(f_2)}{\partial\rho}\right\}$$

$$H_\rho = \frac{1}{k_c^2}\left\{\frac{1}{\rho}\frac{\partial(f_3)}{\partial\phi} - \frac{\partial(f_4)}{\partial\rho}\right\}$$

$$H_\phi = \frac{1}{k_c^2}\left\{-\frac{\partial(f_3)}{\partial\rho} - \frac{1}{\rho}\frac{\partial(f_4)}{\partial\phi}\right\}$$

where $f_1 = j\beta E_z, f_2 = j\omega\mu H_z, f_3 = j\omega\varepsilon E_z, f_4 = j\beta H_z$

To summarize, for TE/TM fields, write down the Helmholtz wave equation of the non-zero longitudinal component of the field then solve it. Next, once the longitudinal field components are known, we can obtain the transversal components of the fields using the above relations. The relations are different for Cartesian and Cylindrical coordinate systems. Cartesian coordinate system relations between the longitudinal components of the fields to the transversal components of the fields are suitable for rectangular waveguide analysis. Cylindrical coordinate system relations between the longitudinal components of the fields to the transversal components of the fields are suitable for circular waveguide analysis.

9.2 Coaxial cables

Let us analyze coaxial cables which are widely used as connectors in high frequency devices and applications. An ideal coaxial line consists of two perfect cylindrical conductors (one inner conductor and another outer conductor, see Fig. 9.2 (a)). The space between the two conductors is filled with a dielectric medium. The two conductors are at two different potentials and field configuration for the dominant TEM mode of propagation is also shown in Fig. 9.2 (b) and (c) respectively. Note that the inner conductor is at a potential of V_0 and the outer conductor is at a potential of 0 V. $\Phi(\rho,\phi)$, scalar potential function, can be found from the Laplace's equation of (9.8) in cylindrical coordinates. Let us write down the Laplace equation in the general curvilinear coordinate system first (refer to chapter 1).

$$\nabla^2\Phi = = \frac{1}{s_1 s_2 s_3}\left[\frac{\partial}{\partial a_1}\left(\frac{s_2 s_3}{s_1}\frac{\partial\Phi}{\partial a_1}\right)+\frac{\partial}{\partial a_2}\left(\frac{s_1 s_3}{s_2}\frac{\partial\Phi}{\partial a_2}\right)+\frac{\partial}{\partial a_3}\left(\frac{s_1 s_2}{s_3}\frac{\partial\Phi}{\partial a_3}\right)\right]$$

For cylindrical coordinate system, scale factors are

$s_1=1, s_2=\rho$ and $s_3=1$

and the three variables are

$$a_1=\rho, a_2=\phi, a_3=z$$

Hence,

$$\nabla^2\Phi = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial\Phi}{\partial z}\right)\right]$$

Besides, in the Laplace equation of (9.8), we require only the transversal (ρ,ϕ) components, longitudinal (z) components are zero. Note that the wave is propagating along the z-axis, that's why the transversal plane is the $\rho\phi-$ plane.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\Phi(\rho,\phi)}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi(\rho,\phi)}{\partial\rho^2}=0$$

The boundary conditions on the two metallic walls (inner and outer conductors) are

$$\Phi(a,\phi)=V_0, \Phi(b,\phi)=0$$

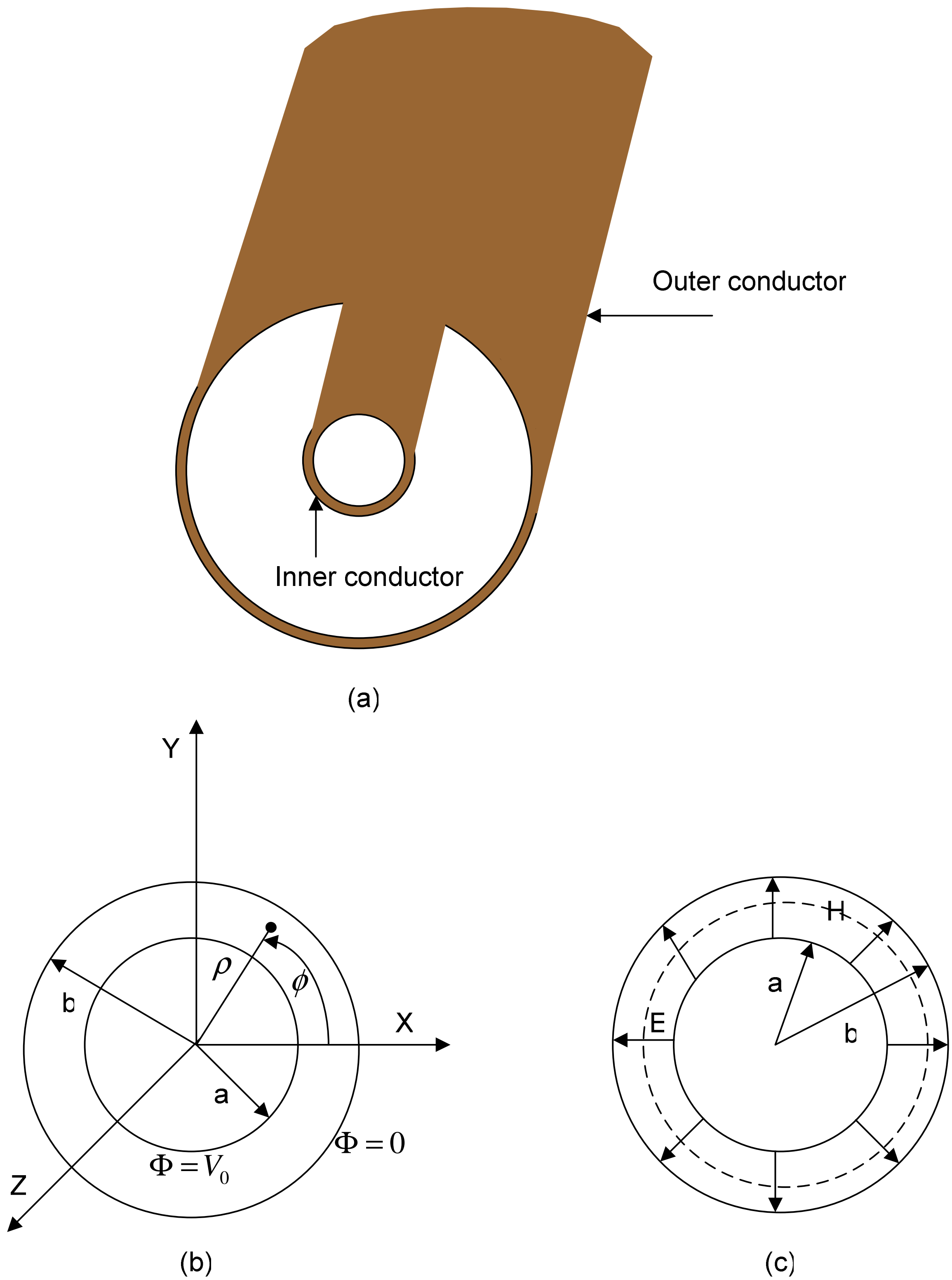


Fig. 9.2 (a) 3-D geometry (b) front view and (c) field distribution of TEM waves in a coaxial cable

A point to be noted here is that in all the analysis we do in this chapter, we will use method of separation of variables. Assumption is made that all readers are familiar with this method. Using the method of separation of variables,

$$\Phi(\rho,\phi) = R(\rho)P(\phi)$$

where R is a function of ρ and P is a function of ϕ only.

Now the above Laplace's equation reduces to

$$\frac{P}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho}+\frac{R}{\rho^2}\frac{\partial^2 P}{\partial\phi^2}=0$$

Multiplying by $\frac{\rho^2}{PR}$, we get,

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho}+\frac{1}{P}\frac{\partial^2 P}{\partial\phi^2}=0$$

Each term must be equal to a constant.

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho}=-k_\rho^2 \qquad (9.17a)$$

$$\frac{1}{P}\frac{\partial^2 P}{\partial\phi^2}=-k_\phi^2 \qquad (9.17b)$$

And $k_\rho^2+k_\phi^2=0$.

General solution of equation 9.17b is

$$P(\phi)=A\cos k_\phi\phi+B\sin k_\phi\phi$$

k_ϕ is an integer then it is single valued since it repeats its value after every cycle of $\phi = 2\pi$. If it is a fraction say ½, the period is 4π, so it will be multiple valued since it involves twice rotation on the coaxial cable circumference. Since the boundary conditions do not vary with ϕ, the potential $\Phi(\phi)$ should not change with ϕ. This implies that k_ϕ must be zero. It implies that $k_\rho = 0$.

$$\Rightarrow \frac{\rho}{R}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} = 0 \Rightarrow \frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} = 0 \Rightarrow \rho\frac{\partial R}{\partial\rho} = C \Rightarrow \frac{\partial R}{\partial\rho} = \frac{C}{\rho} \Rightarrow R = C\ln\rho + D$$

Note that C and D are two unknown constants; we need to find applying boundary conditions.

$$\Phi(a,\phi) = V_0 = C\ln a + D$$
$$\Phi(b,\phi) = C\ln b + D = 0$$
$$\Rightarrow D = -C\ln b$$
$$\therefore V_0 = C\ln a - C\ln b = C\ln\frac{a}{b}$$
$$\Rightarrow C = \frac{V_0}{\ln\frac{a}{b}}; D = -\frac{V_0}{\ln\frac{a}{b}}\ln b$$

$$\Rightarrow \Phi(\rho,\phi) = C\ln\rho + D = \frac{V_0}{\ln\frac{a}{b}}\ln\rho - \frac{V_0}{\ln\frac{a}{b}}\ln b = \frac{V_0\ln\frac{\rho}{b}}{\ln\frac{a}{b}} = \frac{V_0\ln(\frac{b}{\rho})}{\ln(\frac{b}{a})}$$

$$\therefore \vec{e}_t(\rho,\phi) = -\nabla_t\Phi(\rho,\phi)$$

In general curvilinear coordinate system (refer to chapter 1), gradient of a scalar function is defined as

$$\nabla\Phi = \hat{a}_1\frac{\partial\Phi}{s_1\partial a_1} + \hat{a}_2\frac{\partial\Phi}{s_2\partial a_2} + \hat{a}_3\frac{\partial\Phi}{s_3\partial a_3}$$

For cylindrical case and considering transversal components only we have,

$$\therefore \vec{e}_t(\rho,\phi) = -\nabla_t \Phi(\rho,\phi) = -\left(\frac{\partial}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{\phi}\right)\Phi(\rho,\phi) = -\frac{\partial \Phi(\rho,\phi)}{\partial \rho}\hat{\rho}$$

$$= \frac{\partial\left(\dfrac{V_0 \ln(\frac{\rho}{b})}{\ln(\frac{b}{a})}\right)}{\partial \rho}\hat{\rho} = \frac{V_0}{\rho \ln(\frac{b}{a})}\hat{\rho}$$

$$\Rightarrow \vec{E}_t(\rho,\phi) = \frac{V_0 e^{-j\beta z}}{\rho \ln(\frac{b}{a})}\hat{\rho}$$

For TEM waves, magnetic field is perpendicular to both the electric field and the direction of the wave propagation. Its amplitude is reduced by a factor of 1/η from the electric field amplitude. Hence,

$$\vec{h}_t(\rho,\phi) = \frac{1}{\eta}\hat{z}\times\vec{e}_t(\rho,\phi) = \frac{V_0}{\eta\rho \ln(\frac{b}{a})}\hat{\phi}$$

$$\Rightarrow \vec{H}_t(\rho,\phi) = \frac{V_0 e^{-j\beta z}}{\eta\rho \ln(\frac{b}{a})}\hat{\phi}$$

The characteristic impedance of the coaxial line is given by $Z_0 = \dfrac{V(z)}{I(z)}$.

The voltage between the two conductors can be calculated as

$$V(z) = \int_{\rho=a}^{b} \frac{V_0 e^{-j\beta z}}{\rho \ln(\frac{b}{a})} d\rho = V_0 e^{-j\beta z}$$

The total current on the inner conductor at $\rho = a$ can be calculated as

$$I(z) = \int_{\phi=0}^{2\pi} \vec{H} a\, d\phi = \int_{\phi=0}^{2\pi} \frac{V_0 e^{-j\beta z}}{\eta a \ln(\frac{b}{a})} a\, d\phi = \frac{2\pi V_0 e^{-j\beta z}}{\eta \ln(\frac{b}{a})}$$

Note that outer conductor is grounded and in between the two conductors, we usually have some dielectric filling. Besides, the current and voltage are waves propagating along the z-axis.

$$\therefore Z_0 = \frac{V(z)}{I(z)} = \frac{\eta}{2\pi}\ln(\frac{b}{a}) = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{2\pi}\ln(\frac{b}{a})$$

This means depending on the dielectric between the two conductors and ratio of the radius of the outer and inner conductors, we can have coaxial cable of various characteristic impedances. Note that the flow of power in a transmission line takes place entirely via the electric and magnetic fields between the two conductors; power is not transmitted through the conductors themselves. For high power transmission, coaxial cables are used up to 3GHz whereas for low signal transmission they can be used up to 18GHz. In this case, wave travels at the speed of light (non-dispersive). Hence, the phase constant k =β = ω/c. Now that we have found out the two important parameters of a transmission line, let us discuss the issue of higher order modes inside coaxial cables.

Higher order modes:

The basic mode which will be propagating in a coaxial cable is TEM mode. At sufficiently high frequency, some other higher order modes are generated. The lowest higher order mode in coaxial lines is TE_{11} and the cut-off wavelength of these modes is: $\lambda_c(TE_{11}) \cong \pi(a+b) \Rightarrow f_c = \frac{v_p}{\lambda_c} \cong \frac{c}{\sqrt{\mu_r\varepsilon_r}\,2\pi\frac{(a+b)}{2}}$. Therefore, the average circumference of the inner and outer conductors of the coaxial line should be less than the operating wavelength in order to prevent the higher order mode interference. Similar kind of phenomenon occurs in waveguide also, this will be clearer when we discuss overmoded waveguides.

Review Question 9.1: What are TEM, TE and TM waves?

Review Question 9.2: Write down the two source-free Maxwell's curl equations.

Review Question 9.3: Derive the wave equations for TEM, TE and TM waves.

Review Question 9.4: Does electric field of TEM waves satisfy Laplace's equations?

Review Question 9.5: How to obtain the transverse components of electric and magnetic fields (E_x, E_y, H_x and H_y) when the longitudinal components (E_z and H_z) are given in Cartesian coordinate systems?

Review Question 9.6: How to obtain the transverse components of electric and magnetic fields (E_ρ, E_ϕ, H_ρ and H_ϕ) when the longitudinal components (E_z and H_z) are given in Cylindrical coordinate systems?

Review Question 9.7: Write down the expression for propagation constant and characteristic impedance of a coaxial cable.

Review Question 9.8: What is the dominant mode of propagation in a coaxial cable?

Review Question 9.9: How to suppress the propagation of higher order modes in coaxial cable?

9.3 Rectangular waveguide

What are waveguides?

These are hollow metal pipes that guide EM waves. Depending on the shape of the metal pipe: they can be rectangular or circular waveguide. Fiber optical cables are an example of waveguide which operates at optical frequencies. We will assume that waveguide is invariant along z-axis and wave is propagating along the positive z-axis (we may also assume wave propagation along negative z-axis without loss of generality).

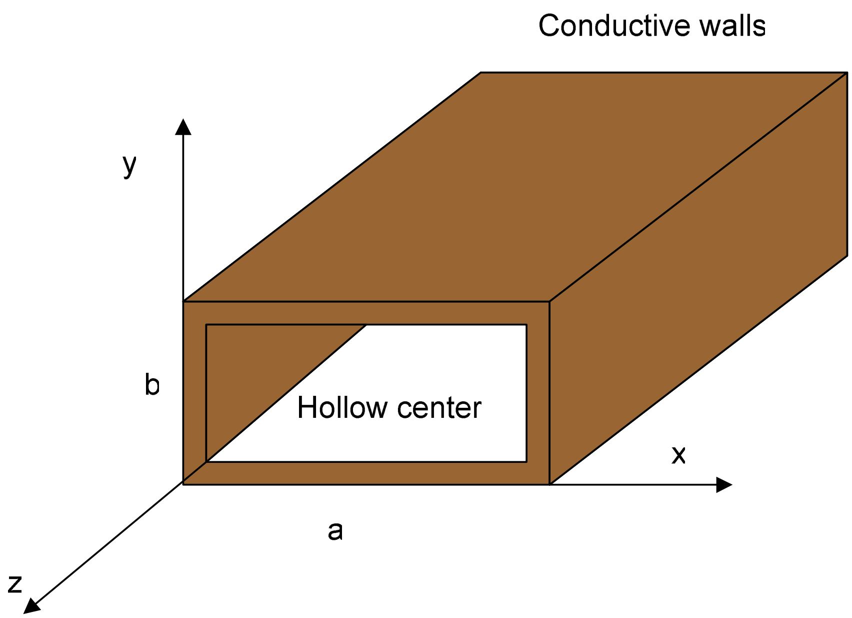


Fig. 9.3 Rectangular waveguide (a and b dimensions are for inner walls of the waveguide)

The geometry of a rectangular waveguide is depicted in Fig. 9.3. It is conventional to have a>b and usually waveguides are hollow metal pipe or filled with a dielectric medium. Since the waveguide has only single conductor, it can't support TEM waves. TE/TM modes will propagate inside a waveguide structure.

9.3.1 TE modes:

For transverse electric or TE modes, $E_z = 0$ and $H_z \neq 0$, since the electric field is transverse or perpendicular to the direction of propagation of the wave. In this case, we have assumed that the waves are propagating along the z-axis. Since $E_z = 0$ and $H_z \neq 0$, we should first find H_z and find the other components of the fields. For propagation in the positive z direction, $H_z = h_z(x,y)e^{-j\beta z}$. Then the equation (9.9) becomes,

$$\therefore \frac{\partial^2 h_z}{\partial x^2} + \frac{\partial^2 h_z}{\partial y^2} = -k_c^2 h_z$$

Applying method of separation of variables, we get,

$$h_z(x,y) = X(x)Y(y)$$

$$\Rightarrow Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = -k_c^2 XY$$

$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_c^2$$

Note that the first term in the above equation is totally dependent on x variable whereas the second term is totally dependent on the variable y and in the RHS of the equation we have a constant. So we can equate the first term to a constant $-k_x^2$ and second term to a constant $-k_y^2$. Then the separation equation becomes

$$\Rightarrow k_x^2 + k_y^2 = k_c^2$$

$\therefore X = A\cos k_x x + B\sin k_x x$ and $Y = C\cos k_y y + D\sin k_y y$.

Hence,

$$H_z(x,y,z) = h_z(x,y)e^{-j\beta z} = X(x)Y(y)e^{-j\beta z} = (A\cos k_x x + B\sin k_x x)\times(C\cos k_y y + D\sin k_y y)e^{-j\beta z}$$

Applying the boundary conditions on the waveguide walls (tangential components of the electric field is zero on a perfect metal, perfect metal was discussed in chapter 3), we have,

$e_x(x,y) = 0$ at y=0,b (horizontal walls) where $E_x(x,y,z) = e_x(x,y)e^{-j\beta z}$

$e_y(x,y)=0$ at x=0,a (vertical walls) where $E_y(x,y,z)=e_y(x,y)e^{-j\beta z}$

From the first boundary condition (using equation 9.13c), we have,

$$E_x\Big|_{y=0,b}$$
$$=\frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}=\frac{-j\omega\mu k_y}{k_c^2}e^{-j\beta z}(A\cos k_x x+B\sin k_x x)(-C\sin k_y y+D\cos k_y y)\Big|_{y=o,b}$$
$$=0$$

$$\Rightarrow D=0, C\sin k_y b=0\Rightarrow k_y b=n\pi$$

From the second boundary condition (using equation 9.13d), we have,

$$E_y\Big|_{x=0,a}$$
$$=\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}=\frac{j\omega\mu}{k_c^2}e^{-j\beta z}k_x(-A sin k_x x+B\cos k_x x)(C\cos k_y y+D\sin k_y y)\Big|_{x=o,a}$$
$$=0$$

$$\Rightarrow B=0, A\sin k_x a=0\Rightarrow k_x a=m\pi$$

Hence, we can write,

$$H_z(x,y,z)=e^{-j\beta z}A\cos k_x xC\cos k_y y=H_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta z}\qquad \text{[Setting A C=}H_0\text{]}$$

$$E_x=\frac{j\omega\mu}{k_c^2}k_y H_0\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta z}$$

$$E_y=\frac{-j\omega\mu}{k_c^2}k_x H_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta z}$$

From equation (9.13a),

$$H_x=\frac{-j}{k_c^2}\beta\frac{\partial H_z}{\partial x}=\frac{j\beta}{k_c^2}k_x H_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta z}$$

From equation (9.13b),

$$H_y = \frac{-j}{k_c^2}\beta\frac{\partial H_z}{\partial x} = \frac{j\beta}{k_c^2}k_y H_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta z}$$

$$where \qquad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}$$

the propagation constant is $\beta = \sqrt{k^2 - k_c^2} = \sqrt{(k^2 - k_x^2 - k_y^2)} = \sqrt{k^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}$

β is real (≥ 0) for a propagation mode. For finding the cut-off frequency, β=0.

$$k = k_c \Rightarrow \omega\sqrt{\mu\varepsilon} = \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2} \Rightarrow f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$$

The mode with the lowest cutoff frequency is called the dominant mode since it is customary to choose the waveguide dimension a>b, so the dominant mode is TE_{10} with cut-off frequency of $f_{c_{10}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\pi}{a} = \frac{1}{2a\sqrt{\mu\varepsilon}}$. The field expressions for $\vec{E}$ and $\vec{H}$ are zero (except H_z) if m=n=0, hence, there are no TE_{00} mode propagating inside the rectangular waveguide. At a given frequency only those modes with $f > f_c$ will propagate, modes with $f < f_c$ will lead to imaginary value of β which implies real value of α, hence these fields decay exponentially away from the source of excitations and such modes are said to be evanescent modes.

Example 9.1

What is an overmoded waveguide? Explain with the help of an X-band waveguide whose dimension is a=22.86 mm and b=10.16 mm.

Solution: In an overmoded waveguide, several modes will be propagating. It is an unwanted situation because there will be power allocation to different modes which are propagating and the analysis of such an overmoded waveguide becomes highly complicated. Let us consider simple hollow X-band waveguide whose dimension is a=22.86 mm and b=10.16 mm. Let us calculate the cut-off frequencies for the first three propagating modes (TE_{10}, TE_{20} and TE_{01}) inside this rectangular waveguide.

$$f_c = \frac{c}{2\pi}\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$$

Note that for TE_{10} mode m=1,n=0; for TE_{20} mode m=2,n=0 and for TE_{01} mode m=0,n=1. Hence the cut-off frequencies for the three modes are 6.56 GHz, 13.12 GHz and 14.7 GHz respectively. Note that for single mode operation of the waveguide for the dominant TE_{10} mode, the frequency range is from 6.56 GHz to 13.12 GHz. This is useful frequency region for the X-band waveguide. Besides, at 14.7GHz, waveguide will have three modes (TE_{10}, TE_{20} and TE_{01}) propagating and we call such waveguides as overmoded waveguide.

The wave impedance that relates the transverse electrical and magnetic fields is given by $Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta}$. Note that Z_{TE} is real for propagating mode since β is real.

Guided wavelength is defined as $\lambda_g = \frac{2\pi}{\beta}$ and phase velocity is defined as $\frac{\omega}{\beta}$.

$$v_p = \frac{\omega}{\sqrt{k^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}}$$

Note that c=ω/k. But for the phase velocity inside the waveguide, denominator is a smaller number than k. Hence, the phase velocity is greater than the speed of light. This do not violate Einstein's law since energy and information flow is dependent on the group velocity.

Example 9.2

Find the group velocity inside rectangular waveguide. Show that it is lesser than the speed of light.

Solution:

Group velocity is defined as

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{\frac{d\beta}{d\omega}} = \frac{1}{\frac{d\left(\sqrt{k^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}\right)}{d\omega}}$$

$$= \frac{1}{\frac{d\left(\sqrt{\left(\frac{\omega}{c}\right)^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}\right)}{d\omega}} = \frac{1}{\frac{1}{2(c)^2}\frac{2\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}}}$$

$$= \frac{(c)^2 \beta}{\omega} = \frac{(c)^2}{v_p}$$

Hence the group velocity is less than the speed of light since phase velocity is greater than the speed of light.

Let us spend few lines on the guided wavelength of the different modes propagating inside the rectangular waveguide. We know that the free space wavelength is given by $\lambda=2\pi/k$. Inside the waveguide, the guided wavelength will be

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}} > \lambda$$

Therefore, different modes will have different guided wavelength and it is longer than the free space wavelength.

9.3.2 Dominant mode of a rectangular waveguide:

For TE_{10} mode, m=1, n=0,

$E_x = 0$

$E_z = 0$

$$E_y = \frac{-j\omega\mu a}{\pi} H_0 \sin\frac{\pi x}{a} e^{-j\beta z}$$

$$H_x = \frac{j\beta a}{\pi} H_0 \sin\frac{\pi x}{a} e^{-j\beta z}$$

$H_y = 0$ and

$$H_z = H_0 \cos\frac{\pi x}{a} e^{-j\beta z}$$

where $\beta = \sqrt{k^2 - (\frac{\pi}{a})^2}$ and $k_c = \sqrt{k_x^2 + k_y^2} = \frac{\pi}{a}$.

The other parameters of interest are:

$$f_c = \frac{c\pi}{2\pi a} = \frac{c}{2a}$$

$$Z_{TE} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\sqrt{k^2 - (\frac{\pi}{a})^2}}$$

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - (\frac{\pi}{a})^2}} > \lambda$$

$$v_p = \frac{\omega}{\sqrt{k^2 - (\frac{\pi}{a})^2}}$$

$$v_g = \frac{(c)^2}{v_p}$$

We can also calculate the power flow through rectangular waveguide for TE_{10} mode as follows:

$$P_{10} = \frac{1}{2}\text{Re}\left\{\int_{x=0}^{a}\int_{y=0}^{b} \vec{E}\times\vec{H}^{*}\bullet\hat{z}dxdy\right\} = \frac{1}{2}\text{Re}\left\{\int_{x=0}^{a}\int_{y=0}^{b} (E_y H_z^{*}\hat{x} - E_y H_x^{*}\hat{z})\bullet\hat{z}dxdy\right\}$$
$$= \frac{1}{2}\beta\frac{\omega\mu a^2}{\pi^2}|H_0|^2\int_{x=0}^{a}\int_{y=0}^{b}\sin^2\frac{\pi x}{a}dxdy = \frac{1}{2}\frac{\omega\mu a^2 b}{\pi^2}|H_0|^2\frac{a}{2}\beta = \frac{1}{4}\frac{\omega\mu a^3 b}{\pi^2}|H_0|^2\beta$$

Note that β is real for TE_{10} mode.

9.3.3 TM modes:

For transverse magnetic or TM modes, $H_z = 0$ and $E_z \neq 0$, since the magnetic field is transverse or perpendicular to the direction of propagation of the wave. In this case, we have assumed that the waves are propagating along the z-axis. Since $H_z = 0$ and $E_z \neq 0$, we should first find E_z and find the other components of the fields. For propagation in the positive z direction, $E_z = e_z(x, y)e^{-j\beta z}$. Then the equation (9.10) becomes,

$$\therefore \frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_z}{\partial y^2} = -k_c^2 e_z$$

Applying method of separation of variables, we get,

$$e_z(x, y) = X(x)Y(y)$$

$$\Rightarrow Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = -k_c^2 XY$$
$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_c^2$$

Note that the first term in the above equation is totally dependent on x variable whereas the second term is totally dependent on the variable y and in the RHS of the equation we have a constant. So we can equate the first term to a constant $-k_x^2$ and second term to a constant $-k_y^2$.

$\Rightarrow k_x^2 + k_y^2 = k_c^2$

$\therefore X = A\cos k_x x + B\sin k_x x$ and $Y = C\cos k_y y + D\sin k_y y$.

Hence,

$$E_z(x,y,z) = e_z(x,y)e^{-j\beta z} = X(x)Y(y)e^{-j\beta z} = (A\cos k_x x + B\sin k_x x)\times(C\cos k_y y + D\sin k_y y)e^{-j\beta z}$$

Applying the boundary conditions on the waveguide walls (tangential components of the electric field is zero), we have,

$e_x(x,y) = 0$ at y=0,b where $E_x(x,y,z) = e_x(x,y)e^{-j\beta z}$

$e_y(x,y) = 0$ at x=0,a $E_y(x,y,z) = e_y(x,y)e^{-j\beta z}$

From the first boundary condition (using equation 9.13c), we have,

$$E_x\Big|_{y=0,b}$$
$$= \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x} = \frac{-j\beta k_x}{k_c^2}e^{-j\beta z}(-A\sin k_x x + B\cos k_x x)(C\cos k_y y + D\sin k_y y)\Big|_{y=o,b}$$
$$= 0$$

$\Rightarrow C = 0, D\sin k_y b = 0 \Rightarrow k_y b = n\pi$

From the second boundary condition (using equation 9.13d), we have,

$$E_y\Big|_{x=0,a}$$
$$= -\frac{j\beta}{k_c^2}\frac{\partial E_z}{\partial y} = -\frac{j\beta k_y}{k_c^2}e^{-j\beta z}(A\cos k_x x + B\sin k_x x)(-C\sin k_y y + D\cos k_y y)\Big|_{x=o,a}$$
$$= 0$$

$\Rightarrow A = 0, B\sin k_x a = 0 \Rightarrow k_x a = m\pi$

Hence, we can write,

$$E_z(x,y,z) = e^{-j\beta z}B\sin k_x x D\sin k_y y = E_0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta z} \qquad \text{[Setting B D=} E_0\text{]}$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x} = \frac{-j\beta k_x}{k_c^2} E_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{-j\beta z}$$

$$E_y = -\frac{j\beta}{k_c^2}\frac{\partial E_z}{\partial y} = -\frac{j\beta k_y}{k_c^2} E_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} e^{-j\beta z}$$

From equation (9.13a),

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{j\omega\varepsilon}{k_c^2} k_y E_0 \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} e^{-j\beta z}$$

From equation (9.13b),

$$H_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x} = -\frac{j\omega\varepsilon}{k_c^2} k_x E_0 \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{-j\beta z}$$

$$\text{where} \qquad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}$$

the propagation constant is $\beta = \sqrt{k^2 - k_c^2} = \sqrt{(k^2 - k_x^2 - k_y^2)} = \sqrt{k^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}$

β is real (≥ 0) for a propagation mode. For finding the cut-off frequency, $\beta=0$.

$$k = k_c \Rightarrow \omega\sqrt{\mu\varepsilon} = \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2} \Rightarrow f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$$

The field expression for E_z is zero if either (or both) m and n are zero, hence, there are no $TM_{00}, TM_{01}, TM_{10}$ modes propagating inside the rectangular waveguide. The first mode which will propagate inside rectangular waveguide is TM_{11} with a cut-off frequency of $f_{c_{11}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2}$.

Let us summarize what we have learnt in rectangular waveguides. There are two sets of waveguide modes, TE and TM modes that can be guided along a rectangular waveguide.

For TE modes, $E_z = 0$ and $H_z(x, y, z) = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta z}$. The other field components can be derived using equation (9.13). The boundary conditions require that $E_x = 0$ at y = 0 and y = b (horizontal walls) and $E_y = 0$ at x = 0 and x = a (vertical walls). Hence, two sets of guidance conditions are derived as $k_x = m\pi/a$ and $k_y = n\pi/b$, where m, n = 0, 1, 2, · · ·. For TE_{00} cases, no field exist inside the rectangular waveguide except $H_z(x, y, z)$ and hence no waveguide mode exist for this case. The TE mode with $k_x = m\pi/a$ and $k_y = n\pi/b$ is denoted as the TE_{mn} mode. Most practical rectangular waveguides operate in the TE_{10} mode also known as the dominant mode with the electric field $E_y = \frac{-j\omega\mu a}{\pi}H_0 \sin\frac{\pi x}{a}e^{-j\beta z}$ where $\beta = \sqrt{k^2 - (\frac{\pi}{a})^2}$, the cutoff wave number is $k_c = \sqrt{k_x^2 + k_y^2} = \frac{\pi}{a}$ and the cutoff wavelength is $\lambda_c = 2a$. Similarly, the TM modes can be derived by setting $H_z = 0$ and $E_z(x, y, z) = E_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta z}$. The other field components can be derived using equation (9.13). The same boundary conditions require that $E_x = 0$ at y = 0 and y = b (horizontal walls) and $E_y = 0$ at x = 0 and x = a (vertical walls). The same guidance conditions are derived as $k_x = m\pi/a$ and $k_y = n\pi/b$ except that *m, n* = 1, 2, · · ·. For TM_{0n}, TM_{m0} and TM_{00} cases, it will give $E_z(x, y, z)$=0 and hence no field exist inside the rectangular waveguide. The TM mode with $k_x = m\pi/a$ and $k_y = n\pi/b$ is denoted as the TM_{mn} mode. The first propagating mode is TM_{11} mode with a cut-off frequency of $f_{c_{11}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2}$.

Review question 9.10: Is the β and Z_{TE} real for propagation of TE modes inside rectangular waveguide?

Review question 9.11: What is the dominant mode of propagation inside rectangular waveguide?

Review question 9.12: What are the cut-off frequencies for TE_{10} and TE_{11} modes inside a rectangular waveguide?

Review question 9.13: Can rectangular waveguide have TE_{00}, TM_{00}, TM_{01} and TM_{10} modes of propagation?

Review question 9.14: What are the field expressions for TE_{10} dominant mode of propagation of a rectangular waveguide?

Review question 9.15: What are evanescent modes inside rectangular waveguide?

Review question 9.16: Is it true that for evanescent modes inside rectangular waveguide β is real?

Review question 9.17: What is an overmoded waveguide? Why don't we want a waveguide to operate in overmoded propagation?

Review question 9.18: Can phase velocity inside waveguide be greater than the speed of light? Does it violate the Einstein's law?

Review question 9.19: What is the product of the phase velocity and group velocity inside rectangular waveguide?

9.4 Circular Waveguide

For hollow circular waveguide of radius a, we could also do the same analysis in cylindrical coordinate system to get TE modes fields propagating inside the cylindrical waveguide depicted in Fig. 9.4.

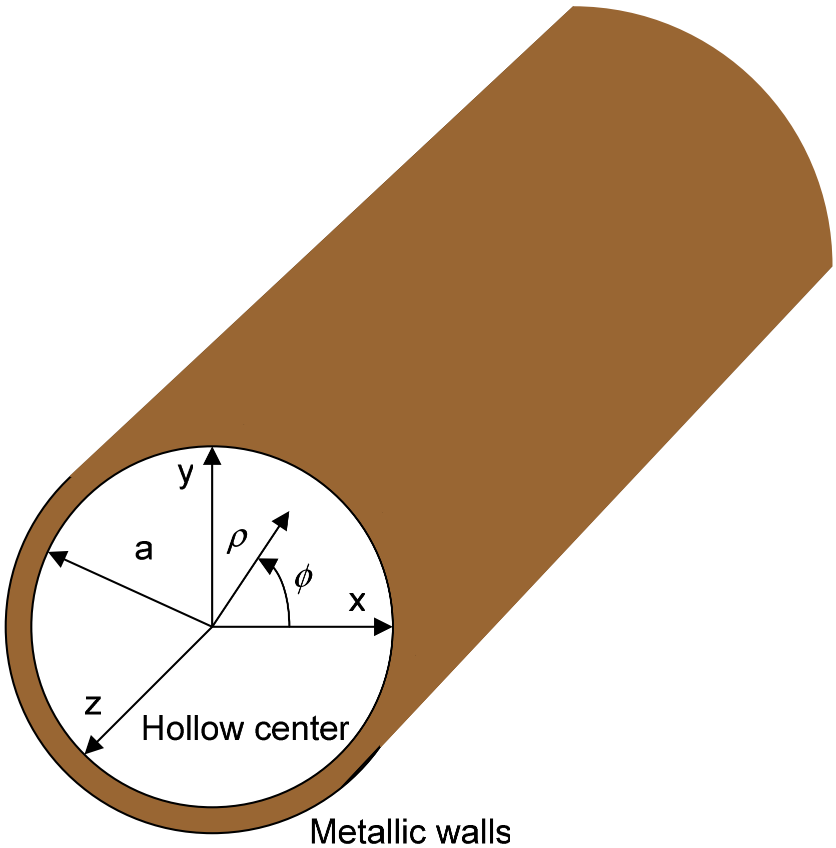


Fig. 9.4 Circular waveguide

9.4.1 TE modes:

As we know that for TE modes with wave propagation along the z-axis, we have, $E_z = 0$ and $H_z \neq 0$. Therefore, the wave equation for H_z is

$\nabla^2 H_z + k^2 H_z = 0$ where $k = \omega\sqrt{\mu\varepsilon}$

For propagation along z-axis, h_z is a function of (ρ, ϕ) only. Mathematically,

$H_z(\rho, \phi, z) = h_z(\rho, \phi) e^{-j\beta z}$

From chapter 1, the Laplacian of a scalar function ϕ in general curvilinear coordinate system is defined as

$$\nabla^2\phi = \frac{1}{s_1 s_2 s_3}\left(\frac{\partial}{\partial a_1}\frac{s_2 s_3}{s_1}\frac{\partial\phi}{\partial a_1} + \frac{\partial}{\partial a_2}\frac{s_1 s_3}{s_2}\frac{\partial\phi}{\partial a_2} + \frac{\partial}{\partial a_3}\frac{s_1 s_2}{s_3}\frac{\partial\phi}{\partial a_3}\right)$$

For cylindrical coordinate system it reduces to

$$\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho\frac{\partial H_z}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2 H_z}{\partial\phi^2} + \rho\frac{\partial^2 H_z}{\partial z^2}\right) + k^2 H_z = 0$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial H_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial\phi^2} + \frac{\partial^2 H_z}{\partial z^2} + k^2 H_z = 0$$

$$\Rightarrow \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \beta^2 + k^2\right)h_z = 0$$

$$\Rightarrow \left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_c^2\right)h_z = 0$$

where $k_c^2 = k^2 - \beta^2$

Applying the method of separation of variables, let us assume that

$$h_z(\rho,\phi) = R(\rho)P(\phi)$$

Hence the above equation reduces to

$$\frac{P}{\rho}\frac{\partial R}{\partial\rho} + P\frac{\partial^2 R}{\partial\rho^2} + \frac{R}{\rho^2}\frac{\partial^2 P}{\partial\phi^2} + k_c^2 RP = 0$$

Now multiply the above equation by $\frac{\rho^2}{RP}$, we get,

$$\Rightarrow \frac{\rho}{R}\frac{\partial R}{\partial\rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{P}\frac{\partial^2 P}{\partial\phi^2} + k_c^2\rho^2 = 0$$

Equating the 3rd term in the above equation to a constant $-k_\phi^2$, we have,

$$\frac{1}{P}\frac{\partial^2 P}{\partial \phi^2} = -k_\phi^2 \Rightarrow P = A\cos k_\phi \phi + B\sin k_\phi \phi$$

k_ϕ must be an integer values otherwise the function P will be multiple valued. Hence,

$$P = A\cos n\phi + B\sin n\phi$$

where n is an integer.

Therefore the main equation

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{P}\frac{\partial^2 P}{\partial \phi^2} + k_c^2\rho^2 = 0$$

can be further simplified as

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} - n^2 + k_c^2\rho^2 = 0$$

$$\Rightarrow \rho\frac{\partial R}{\partial \rho} + \rho^2\frac{\partial^2 R}{\partial \rho^2} + \left(k_c^2\rho^2 - n^2\right)R = 0$$

Solution for the above Bessel's differential equation is

$$R = CJ_n(k_c\rho) + DY_n(k_c\rho)$$

where $J_n(k_c\rho) = \sum_{m=0}^{\infty}\frac{(-1)^m (k_c\frac{\rho}{2})^{2m+n}}{m!(m+n)!}$ is the Bessel's function of first kind of n[th] order and

$Y_n(k_c\rho) = \frac{J_n(k_c\rho)\cos(n\pi) - J_{-n}(k_c\rho)}{\sin n\pi}$ is the Bessel's function of second kind of n^{th} order.

We can also find

$$J_{-n}(k_c\rho) = \sum_{m=0}^{\infty}\frac{(-1)^m (k_c\frac{\rho}{2})^{2m-n}}{m!(m-n)!}$$

Note that $Y_n\left(k_c\rho\right) \to \infty$ as $(k_c\rho) \to 0$ which is physically not acceptable as fields must be finite at the origin and hence choose D=0.

Now, $R = CJ_n(k_c\rho)$.

The expression for the longitudinal component of the magnetic field is $H_z(\rho,\phi,z) = (A\cos n\phi + B\sin n\phi)J_n(k_c\rho)e^{-j\beta z}$

One of the two terms of P(ϕ) can be eliminated. Both have the same field pattern except for a rotation of 90^0 in the ϕ direction. Also the excitation can be chosen in such a way that only one term can exist at any time. Setting B=0, we can further simplify the above expression as

$$H_z(\rho,\phi,z) = (A\cos n\phi)J_n(k_c\rho)e^{-j\beta z}$$

$\because E_\phi = 0$ at the waveguide walls (tangential component of the electric field is zero at the metallic walls), we can apply this boundary condition. From equation 9.16(d),

$$E_\phi\Big|_{\rho=a} = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial\rho}\Bigg|_{\rho=a} = \frac{j\omega\mu k_c}{k_c^2}A\cos n\phi J_n^{'}(k_c\rho)e^{-j\beta z}\Bigg|_{\rho=a} = 0$$

$$\Rightarrow J_n^{'}(k_c\rho)\Big|_{\rho=a} = 0 \Rightarrow k_c a = p_{nm}^{'}$$

The m^{th} root of derivative of the nth order Bessel's function of first kind $J_n^{'}(k_c\rho)$ is denoted by $p_{nm}^{'}$ and $k_c a = p_{nm}^{'}$

A note on Bessel's functions: $J_0(x)$ *and* $J_1(x)$

For n=0,

$$J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + ...$$

which looks like a cosine function.

For n=1,

$$J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}m!(m+1)!}=\frac{x}{2}-\frac{x^3}{2^3 1!2!}+\frac{x^5}{2^5 2!3!}-\frac{x^7}{2^7 3!4!}+\ldots\ldots\ldots\ldots\ldots$$

which looks like a sine function.

But the zeros are not completely regularly spaced and the height of the wave decrease with increasing x.

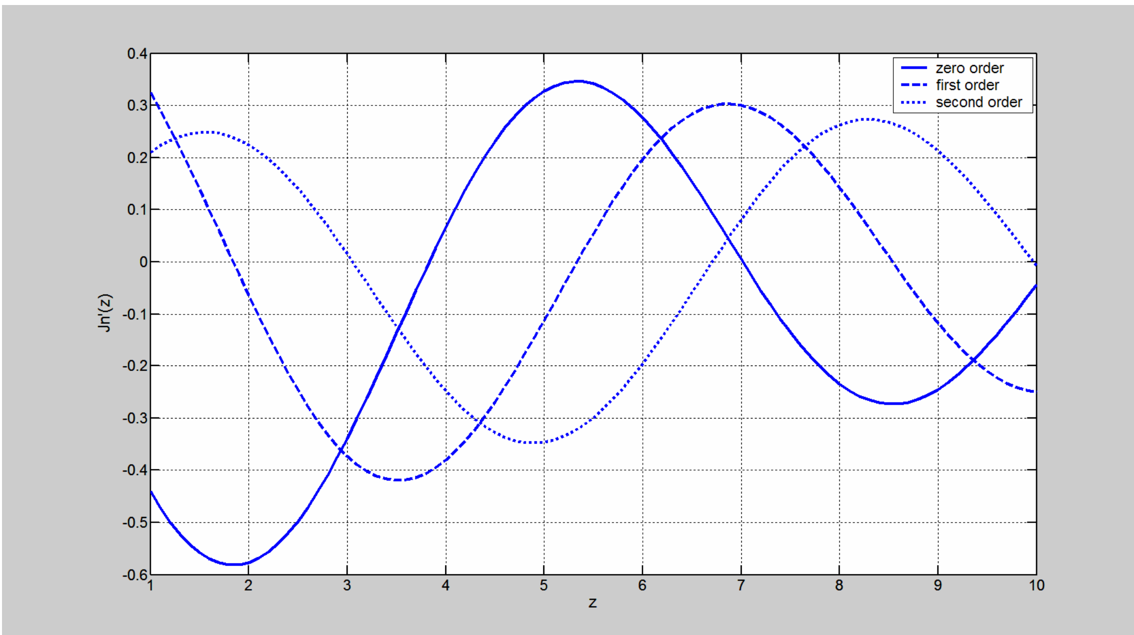


Fig. 9.5 Plot of derivative of Bessel's function of first kind of order 0, 1 and 2 [note that $J_n^{'}(z)=-J_{n+1}(z)+\frac{nJ_n(z)}{z}$ and MATLAB command for $J_n(z)$ is besselj (n, z)]

Table 9.1 Values for $p_{nm}^{'}$ for different values of m (m[th] root of the derivative of the Bessel's function) and n (order of the Bessel's function)

n m	1	2
0	3.832	7.016
1	1.841	5.331

Now, $H_z(\rho,\phi,z) = (A\cos n\phi)J_n(k_c\rho)e^{-j\beta z}$

For other fields components are obtained using equation 9.16.

$$E_\rho = \frac{-j}{k_c^2}\frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial\phi} = \frac{j}{k_c^2}\frac{\omega\mu n}{\rho}J_n(k_c\rho)(A\sin n\phi)e^{-j\beta z}$$

$$E_\phi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial\rho} = \frac{j\omega\mu}{k_c}A\cos n\phi J_n^{'}(k_c\rho)e^{-j\beta z}$$

$$H_\rho = \frac{-j}{k_c^2}\beta\frac{\partial H_z}{\partial\rho} = \frac{-j\beta}{k_c}A\cos n\phi J_n^{'}(k_c\rho)e^{-j\beta z}$$

$$H_\phi = \frac{-j}{k_c^2}\frac{\beta}{\rho}\frac{\partial H_z}{\partial\phi} = \frac{j\beta n}{k_c^2\rho}(A\sin n\phi)J_n(k_c\rho)e^{-j\beta z}$$

where $k_c = \frac{p_{nm}^{'}}{a}$

But what we want is $p_{01}^{'}$, $p_{11}^{'}$ and so on which are listed in the table 9.1 and it can be obtained from Fig. 9.5.

For TE_{nm} mode,

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - (\frac{p_{nm}^{'}}{a})^2}$$

For cutoff frequency, β_{nm}=0

$$\Rightarrow f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{p_{nm}^{'}}{a}$$

The mode with the lowest cutoff frequency is called the dominant mode. Dominant mode is TE_{11} for $p_{11}^{'}$ since it has the smallest value of 1.841 with cut-off frequency of

$f_{c_{11}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{1.841}{a}$. The field expressions for $\vec{E}$ and $\vec{H}$ do not exist for m=n=0, hence, there are no TE_{00} mode propagating inside the circular waveguide.

The wave impedance is $Z_{TE} = \frac{E_\rho}{H_\phi} = \frac{\omega\mu}{\beta}$. Guided wavelength is defined as $\lambda_g = \frac{2\pi}{\beta}$ and phase velocity is defined as $\frac{\omega}{\beta}$.

$$v_p = \frac{\omega}{\sqrt{k^2 - (\frac{p'_{nm}}{a})^2}}$$

Note that c=ω/k. But for the phase velocity inside the waveguide, denominator is a smaller number than k. Hence, the phase velocity is greater than the speed of light. This do not violate Einstein's law since energy and information flow is dependent on the group velocity. Group velocity is defined as

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{\frac{d\beta}{d\omega}} = \frac{1}{\frac{d\left(\sqrt{k^2 - (\frac{p'_{nm}}{a})^2}\right)}{d\omega}}$$

$$= \frac{1}{\frac{d\left(\sqrt{\left(\frac{\omega}{c}\right)^2 - (\frac{p'_{nm}}{a})^2}\right)}{d\omega}} = \frac{1}{\frac{1}{2(c)^2}\frac{2\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - (\frac{p'_{nm}}{a})^2}}}$$

$$= \frac{(c)^2\beta}{\omega} = \frac{(c)^2}{v_p} < c$$

9.4.2 Dominant mode of a circular waveguide:

For the dominant TE_{11} mode, the field expressions are

$$E_\rho = \frac{j\omega\mu}{k_c^2 \rho} J_1(k_c \rho)(A \sin\phi) e^{-j\beta z}$$

$$E_\phi = \frac{j\omega\mu}{k_c} A \cos\phi J_1^{'}(k_c \rho) e^{-j\beta z}$$

$$E_z = 0$$

$$H_\rho = \frac{-j\beta}{k_c} A \cos\phi J_1^{'}(k_c \rho) e^{-j\beta z}$$

$$H_\phi = \frac{j\beta}{k_c^2 \rho} J_1(k_c \rho)(A \sin\phi) e^{-j\beta z}$$

$$H_z = A \cos\phi J_1^{'}(k_c \rho) e^{-j\beta z}$$

where $\beta = \sqrt{k^2 - (\frac{1.841}{a})^2}$ and $k_c = \frac{1.841}{a}$.

The other parameters of interest are:

$$f_c = \frac{1.841c}{2\pi a}$$

$$Z_{TE} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\sqrt{k^2 - (\frac{1.841}{a})^2}}$$

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - (\frac{1.841}{a})^2}} > \lambda$$

$$v_p = \frac{\omega}{\sqrt{k^2 - (\frac{1.841}{a})^2}}$$

$$v_g = \frac{(c)^2}{v_p}$$

9.4.3 TM modes:

As we know that for TM modes with wave propagation along the z-axis, we have, $E_z \neq 0$ and $H_z = 0$. Therefore, the wave equation for E_z is

$\nabla^2 E_z + k^2 E_z = 0$ where $k = \omega\sqrt{\mu\varepsilon}$

For propagation along z-axis, e_z is a function of (ρ, ϕ) only. Mathematically,

$$E_z(\rho,\phi,z) = e_z(\rho,\phi)e^{-j\beta z}$$

For cylindrical coordinate system, wave equation could be written as

$$\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho\frac{\partial E_z}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2 E_z}{\partial\phi^2} + \rho\frac{\partial^2 E_z}{\partial z^2}\right) + k^2 E_z = 0$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial E_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0$$

$$\Rightarrow (\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \beta^2 + k^2)e_z = 0$$

$$\Rightarrow (\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_c^2)e_z = 0$$

where $k_c^2 = k^2 - \beta^2$

Applying the method of separation of variables, let us assume that

$$e_z(\rho,\phi) = R(\rho)P(\phi)$$

Hence the above equation reduces to

$$\frac{P}{\rho}\frac{\partial R}{\partial\rho} + P\frac{\partial^2 R}{\partial\rho^2} + \frac{R}{\rho^2}\frac{\partial^2 P}{\partial\phi^2} + k_c^2 RP = 0$$

Now multiply the above equation by $\frac{\rho^2}{RP}$, we get,

$$\Rightarrow \frac{\rho}{R}\frac{\partial R}{\partial\rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial\rho^2} + \frac{1}{P}\frac{\partial^2 P}{\partial\phi^2} + k_c^2\rho^2 = 0$$

Equating the 3rd term in the above equation to a constant $-k_\phi^2$, we have,

$$\frac{1}{P}\frac{\partial^2 P}{\partial \phi^2} = -k_\phi^2 \Rightarrow P = A\cos k_\phi \phi + B\sin k_\phi \phi$$

k_ϕ must be an integer values otherwise the function P will be multiple valued. Hence,

$$P = A\cos n\phi + B\sin n\phi$$

where n is an integer.

Therefore the main equation

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{P}\frac{\partial^2 P}{\partial \phi^2} + k_c^2\rho^2 = 0$$

can be further simplified as

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} - n^2 + k_c^2\rho^2 = 0$$

$$\Rightarrow \rho\frac{\partial R}{\partial \rho} + \rho^2\frac{\partial^2 R}{\partial \rho^2} + \left(k_c^2\rho^2 - n^2\right)R = 0$$

Solution for the above Bessel's differential equation is

$$R = CJ_n(k_c\rho) + DY_n(k_c\rho)$$

where $J_n(k_c\rho) = \sum_{m=0}^{\infty}\frac{(-1)^m (k_c\frac{\rho}{2})^{2m+n}}{m!(m+n)!}$ is the Bessel's function of first kind of nth order and

$Y_n(k_c\rho) = \frac{J_n(k_c\rho)\cos(n\pi) - J_{-n}(k_c\rho)}{\sin n\pi}$ is the Bessel's function of second kind of n^{th} order.

We can also find

$$J_{-n}(k_c\rho) = \sum_{m=0}^{\infty}\frac{(-1)^m (k_c\frac{\rho}{2})^{2m-n}}{m!(m-n)!}$$

Note that $Y_n \to \infty$ as $(k_c \rho) \to 0$ which is physically not acceptable as fields must be finite at the origin and hence choose D=0.

Now, $R = CJ_n(k_c \rho)$.

The expression for the longitudinal component of the electric field is $E_z(\rho, \phi, z) = (A\cos n\phi + B\sin n\phi)J_n(k_c\rho)e^{-j\beta z}$

One of the two terms of P(ϕ) can be eliminated. Both have the same field pattern except for a rotation of 90^0 in the ϕ direction. Also the excitation can be chosen in such a way that only one term can exist at any time. Setting B=0, we can further simplify the above expression as

$$E_z(\rho, \phi, z) = (A\cos n\phi)J_n(k_c\rho)e^{-j\beta z}$$

$\because E_\phi = 0$ at the waveguide walls (tangential component of the electric field is zero at the metallic walls), we can apply this boundary condition. From equation 9.16 (d),

$$E_\phi\Big|_{\rho=a} = -\frac{j\beta}{k_c^2 \rho}\frac{\partial E_z}{\partial \phi}\Bigg|_{\rho=a} = \frac{j\beta n}{k_c^2 \rho} A\sin n\phi J_n(k_c\rho)e^{-j\beta z}\Bigg|_{\rho=a} = 0$$

$$\Rightarrow J_n(k_c\rho)\Big|_{\rho=a} = 0 \Rightarrow k_c a = p_{nm}$$

The m^{th} zero of $J_n(k_c\rho)$ is denoted by p_{nm} and $k_c a = p_{nm}$.

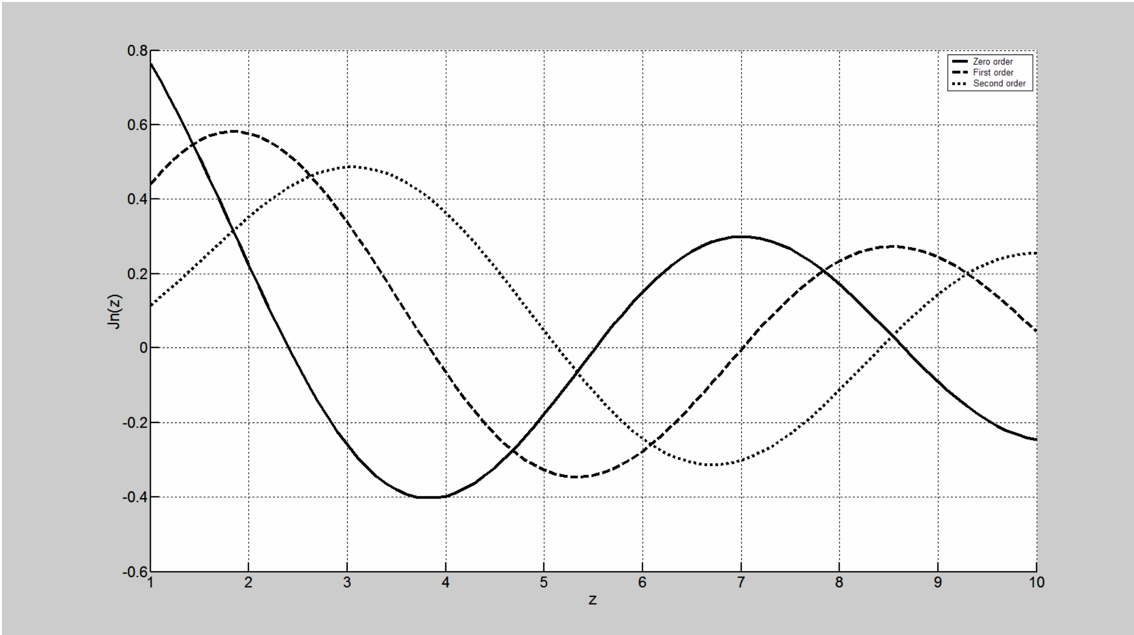


Fig. 9.6 Plot of Bessel's function of first kind of order 0, 1 and 2

Table 9.2 Values for p_{nm} for different values of m (mth zero of the Bessel's function) and n (order of the Bessel's function)

n m	1	2
0	2.405	5.520
1	3.832	7.016

Now, $E_z(\rho,\phi,z)=(A\cos n\phi)J_n(k_c\rho)e^{-j\beta z}$

For other fields components are obtained using equation 9.16.

$$E_\rho=\frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial\rho}=\frac{-j\beta}{k_c}J_n^{'}(k_c\rho)(A\cos n\phi)e^{-j\beta z}$$

$$E_\phi=-\frac{j\beta}{k_c^2\rho}\frac{\partial E_z}{\partial\phi}=\frac{j\beta n}{k_c^2\rho}A\sin n\phi J_n(k_c\rho)e^{-j\beta z}$$

$$H_\rho=\frac{j\omega\varepsilon}{k_c^2\rho}\frac{\partial E_z}{\partial\phi}=-\frac{j\omega\varepsilon n}{k_c^2\rho}A\sin n\phi J_n(k_c\rho)e^{-j\beta z}$$

$$H_\phi = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial \rho} = \frac{-j\omega\varepsilon}{k_c}(A\cos n\phi)J_n^{'}(k_c\rho)e^{-j\beta z}$$

But what we want is p_{01} , p_{11} and so on which are listed in the table 9.2 and they can be obtained from Fig. 9.6.

For TM_{nm} mode,

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - (\frac{p_{nm}}{a})^2}$$

For cutoff frequency, β_{nm} =0

$$\Rightarrow f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{p_{nm}}{a}$$

The TM mode with the lowest cut-off frequency is TM_{01} for p_{01} since it has the smallest value of 2.405 ($f_{c_{01}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{2.405}{a}$). This is not the dominant mode since TE_{11} has lower cut-off frequency than this mode inside circular waveguide. The field expressions for $\vec{E}$ and $\vec{H}$ do not exist for m=n=0, hence, there are no TM_{00} mode propagating inside the circular waveguide.

The wave impedance is $Z_{TE} = \frac{E_\rho}{H_\phi} = \frac{\omega\mu}{\beta}$. Guided wavelength is defined as $\lambda_g = \frac{2\pi}{\beta}$ and phase velocity is defined as $\frac{\omega}{\beta}$.

$$v_p = \frac{\omega}{\sqrt{k^2 - (\frac{p_{nm}}{a})^2}}$$

Note that c=ω/k. But for the phase velocity inside the waveguide, denominator is a smaller number than k. Hence, the phase velocity is greater than the speed of light. This do not Einstein's law since energy and information flow is dependent on the group velocity. Group velocity is defined as

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{\frac{d\beta}{d\omega}} = \frac{1}{\frac{d\left(\sqrt{k^2 - (\frac{p_{nm}}{a})^2}\right)}{d\omega}}$$

$$= \frac{1}{\frac{d\left(\sqrt{\left(\frac{\omega}{c}\right)^2 - (\frac{p_{nm}}{a})^2}\right)}{d\omega}} = \frac{1}{\frac{1}{2(c)^2}\frac{2\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - (\frac{p_{nm}}{a})^2}}}$$

$$= \frac{(c)^2 \beta}{\omega} = \frac{(c)^2}{v_p} < c$$

Dispersion is of major concern when multi-frequency or broadband signals are propagated using a TE or TM mode. These types of signals suffer distortion as they propagate along the structure since different components of the signals propagate at different velocities. TE and TM waves of different frequencies propagate at different velocities and are attenuated at different rates on a waveguiding structure also known as *dispersion*. For instance, the phase velocities of TE/TM waves inside rectangular/circular waveguide is given by

$$\because f = \frac{ck}{2\pi}, f_{c_{mn}} = \frac{ck_c}{2\pi} \therefore \left(\frac{f_{c_{mn}}}{f}\right)^2 = \frac{k_c^2}{k^2}$$

$$\Rightarrow v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{k^2 - k_c^2}} = \frac{\omega}{k\sqrt{1 - \frac{k_c^2}{k^2}}} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

On the other hand, the phase velocity of TEM waves (such as on a lossless or low-loss coaxial cables) are independent of frequency

$$\because \beta = k \therefore v_p = \frac{\omega}{\beta} = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Let us summarize what we have learnt in circular waveguides. There are two sets of waveguide modes, TE and TM modes that can be guided along a circular waveguide. For TE modes, $E_z = 0$ and $H_z(\rho,\phi,z) = (A\cos n\phi)J_n(k_c\rho)e^{-j\beta z}$. The other field components can be derived using equation (9.16). The boundary conditions require that $E_\phi = 0$ on the waveguide walls. Hence, one set of guidance condition is derived as $k_c a = p'_{nm}$, where p'_{nm} denotes m^{th} root of the derivative of the nth order Bessel's function of first kind $J'_n(k_c\rho)$. For TE$_{n0}$ cases, no field exist inside the circular waveguide since there is no 0th root of the $J'_n(k_c\rho)$ and hence m=1,2,…. The TE mode with $k_c a = p'_{nm}$ is denoted as the TE$_{nm}$ mode. Most practical circular waveguides operate in the TE$_{11}$ mode also known as the dominant mode with the electric field $E_\rho = \frac{j\omega\mu}{k_c^2\rho}J_1(k_c\rho)(A\sin\phi)e^{-j\beta z}$ and $E_\phi = \frac{j\omega\mu}{k_c}A\cos\phi J'_1(k_c\rho)e^{-j\beta z}$ where $\beta = \sqrt{k^2 - (\frac{1.841}{a})^2}$, the cutoff wave number is $k_c = \frac{1.841}{a}$ and the cutoff wavelength is $\lambda_c = \frac{2\pi a}{1.841}$. Similarly, the TM modes can be derived by setting $H_z = 0$ and $E_z(\rho,\phi,z) = (A\cos n\phi)J_n(k_c\rho)e^{-j\beta z}$. The other field components can be derived using equation (9.16). The same boundary conditions require that $E_\phi = 0$ on the waveguide walls. The similar guidance condition is derived as $k_c a = p_{nm}$, where p_{nm} is the m^{th} zero of $J_n(k_c\rho)$. For TM$_{n0}$ cases, no field exist inside the rectangular waveguide since there is no 0th zero of the $J_n(k_c\rho)$ and hence m=1,2,…..

The TM mode with $k_c a = p_{nm}$ is denoted as the TM_{nm} mode. The first propagating mode is TM_{01} mode with cut-off frequency of $f_{c_{01}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{2.405}{a}$.

Example 9.3

Find the values of (a) p_{nm} for m=1,2,3 and n=0,1,2 (b) p'_{nm} for m=1,2,3 and n=0,1,2.

Solution:

(a) Let us plot $J_n(z)$ versus z for n=0,1,2 and the find the value of z for first three zero crossings for each $J_n(z)$.

MATLAB program for plotting Fig. 9.6

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all
clc
z=0:0.01:10;
J0=besselj(0,z); %Zero order Bessel function of first kind
J1=besselj(1,z); %First order Bessel function of first kind
J2=besselj(2,z);%Second order Bessel function of first kind
plot(z,J0,'-',z,J1,'--',z,J2,':');
legend('Zero','First','Second');
xlabel('z');
ylabel('Bessel function of first kind')
grid on;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Values p_{nm} for m=1,2 and n=0,1 are given in Table 9.2, other values we can always find from the graph in Fig. 9.6.

(b) Let us plot $J_n^{'}(z)$ versus z for n=0,1,2 and find the value of z for first three zero crossings for each $J_n^{'}(z)$

MATLAB program for plotting Fig. 9.5

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all
clc
z=1:0.01:10;
dJ0=-besselj(1,z); %derivative of Zero order Bessel function of first kind
dJ1=(besselj(1,z))./z-besselj(2,z);%derivative of First order Bessel function of
first kind
dJ2=(2*besselj(2,z))./z-besselj(3,z);%derivative of Second order Bessel function
of first kind
plot(z,dJ0,'-',z,dJ1,'--',z,dJ2,':');
legend('Zero','First','Second');
xlabel('z');
ylabel('Derivative of Bessel function of first kind')
grid on;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Values of $p_{nm}^{'}$ for m=1,2 and n=0,1 are given in Table 9.1 and other values we can always find from the graph in Fig. 9.5.

Review question 9.20: What is the dominant mode of propagation in circular waveguide?

Review question 9.21: Can TE_{00} and TM_{00} mode propagate in a circular waveguide?

Review question 9.22: What is the expression for fc_{nm}, β_{nm} and Z_{nm} for TE modes of propagation inside circular waveguide?

Review question 9.23: What is the expression for fc_{nm}, β_{nm} and Z_{nm} for TM modes of propagation inside circular waveguide?

Review question 9.24: What is the cut-off frequency for a circular waveguide for (a) TM_{01} and (b) TE_{11} modes?

Review question 9.25: What are the field expressions for dominant mode of propagation inside circular waveguide?

Review question 9.26: What is dispersion? Explain in few words.

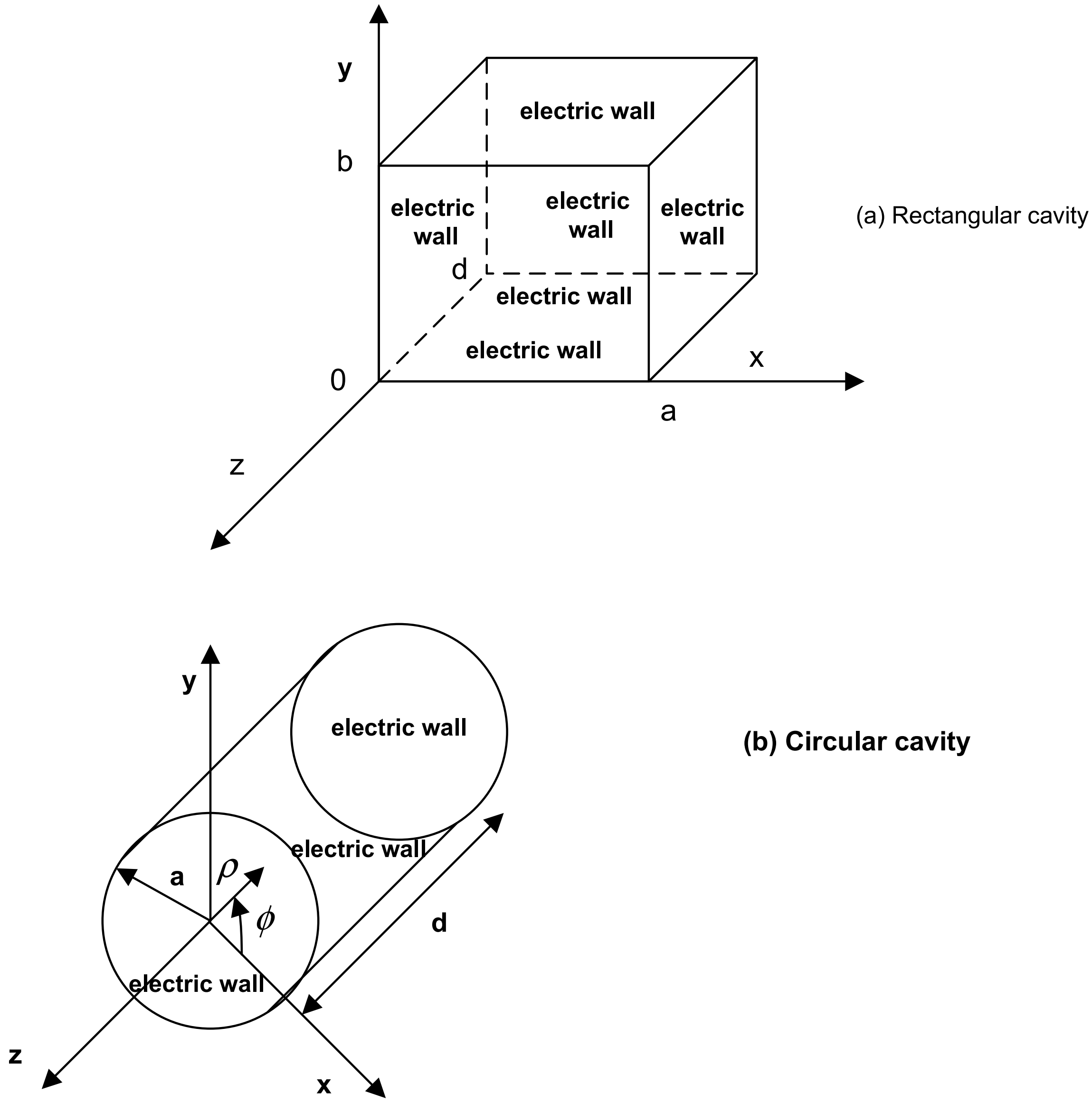


Fig. 9.7 (a) Rectangular cavity (b) Circular cavity

9.5 Rectangular Cavity Resonator

What is this rectangular cavity in first place? Rectangular cavity is basically a rectangular waveguide of finite length (d) whose two end walls are closed with metals. Since we know that perfect metals have reflection coefficient of 1, fields which were propagating along z-axis before putting the metal walls at the two ends of the waveguide, will be totally reflected at the two end walls. This reflected field along with the incident field will form standing wave patterns and starts resonating. This is the basic idea of rectangular cavity resonator. It is basically a closed rectangular metallic box. Then if everything is closed, where the signal does comes from. Usually a small hole is created in the side walls and signals are excited inside the closed metal box using some techniques which is out of the scope of this book. Rectangular cavity is a high quality resonator used for high frequency applications. Fig. 9.7 (a) depicts a rectangular waveguide cavity of dimension (a×b×d). Vector wave equation for charge free region inside rectangular cavity

$$\nabla^2\vec{E}+k^2\vec{E}=0$$

$$\Rightarrow\nabla^2\vec{H}+k^2\vec{H}=0$$

This can be further simplified as

$$\frac{\partial^2\vec{E}}{\partial x^2}+\frac{\partial^2\vec{E}}{\partial y^2}+\frac{\partial^2\vec{E}}{\partial z^2}+k^2\vec{E}=0$$

$$\Rightarrow\frac{\partial^2\vec{H}}{\partial x^2}+\frac{\partial^2\vec{H}}{\partial y^2}+\frac{\partial^2\vec{H}}{\partial z^2}+k^2\vec{H}=0\,.$$

The above equation can be reduced to three wave equations each for H_x, H_y and H_z. Let us consider TE_{mnl} modes for this cavity ($E_z=0, H_z\neq 0$). Hence, we are interested in solving the following wave equation

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+\frac{\partial^2 H_z}{\partial z^2}+k^2H_z=0$$

Applying method of separation of variables (assuming $H_z(x,y,z)=X(x)Y(y)Z(z)$)

$$\therefore\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-\omega^2\varepsilon\mu=-k^2\,.$$

RHS is a constant and each term in the LHS are independent of each other and can be equated to $-k_x^2, -k_y^2, -k_z^2$, respectively. Now the separation equation becomes

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

The solution of X, Y and Z are respectively

$$X = A\cos k_x x + B\sin k_x x \,, Y = C\cos k_y y + D\sin k_y y \,, Z = F\cos k_z z + G\sin k_z z \,.$$

Applying the boundary conditions on the metal walls of the cavity, we have, $E_x(x, y, z)\big|_{y=0,b} = 0$, $E_y(x, y, z)\big|_{x=0,a} = 0$ and $H_z(x, y, z)\big|_{z=0,d} = 0$. The previous two boundary conditions are the same as that of the rectangular waveguides. The last boundary condition is based on the perfect metal boundary condition for magnetic fields as discussed in chapter 4.

The general expression for

$$H_z(x, y, z) = XYZ = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)(F\cos k_z z + G\sin k_z z)$$

Applying the boundary condition on the z-component of the magnetic field, we have,

$$\because H_z(x, y, z)\big|_{z=0,d} = 0 \Rightarrow F = 0, \sin k_z d = 0 \Rightarrow k_z d = l\pi \,.$$

Applying the boundary condition on the x-component of the electric field, we have,

$$E_x(x, y, z) = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} = \frac{-j\omega\mu k_y}{k_c^2}(A\cos k_x x + B\sin k_x x)(D\cos k_y y - C\sin k_y y)(G\sin k_z z)$$

$$E_x(x, y, z)\big|_{y=0,b} = 0 \Rightarrow D = 0 \quad and \qquad \sin k_y b = 0 \Rightarrow k_y b = n\pi$$

Applying the boundary condition on the y-component of the electric field, we have,

$$E_y(x, y, z) = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu k_x}{k_c^2}(B\cos k_x x - A\sin k_x x)(C\cos k_y y)(G\sin k_z z)$$

$$E_y(x, y, z)\big|_{x=0,a} = 0 \Rightarrow B = 0 \quad and \qquad \sin k_x a = 0 \Rightarrow k_x a = m\pi$$

Now we can simplify the above expression for $H_z(x,y,z)$ as follows:

$H_z(x,y,z) = A\cos k_x x C\cos k_y y G\sin k_z z = ACG\cos k_x x\cos k_y y\sin k_z z$

Setting ACG=H_0, we have,

$H_z(x,y,z) = H_0\cos k_x x\cos k_y y\sin k_z z$

where $k_x = \dfrac{m\pi}{a}$, $k_y = \dfrac{n\pi}{b}, k_z = \dfrac{l\pi}{d}$ and m, n, l are the integers.

From the separation equation, $k_x^2 + k_y^2 + k_z^2 = k^2$, we have,

$$f_r = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 + (\frac{l\pi}{d})^2}$$

It is customary to chose d>a>b, then the dominant mode is TE_{101} mode with cut-off frequency

$$f_{r_{101}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{\pi}{a})^2 + (\frac{\pi}{d})^2}$$

Now let us try to find the expressions for all field components inside the rectangular waveguide.

$H_z = H_0\cos k_x x\cos k_y y\sin k_z z$

All the field components can be expressed in term of H_z from equation (9.13).

$$H_x = \frac{j\left(\omega\varepsilon\frac{\partial}{\partial y}E_z - \beta\frac{\partial}{\partial x}H_z\right)}{k_c^2} \tag{9.13a}$$

$$H_y = -\frac{j\left(\omega\varepsilon\frac{\partial}{\partial x}E_z + \beta\frac{\partial}{\partial y}H_z\right)}{k_c^2} \tag{9.13b}$$

$$E_x = -\frac{j\left(\beta \frac{\partial}{\partial x} E_z + \omega\mu \frac{\partial}{\partial y} H_z\right)}{k_c^2} \tag{9.13c}$$

$$E_y = \frac{j\left(-\beta \frac{\partial}{\partial y} E_z + \omega\mu \frac{\partial}{\partial x} H_z\right)}{k_c^2} \tag{9.13d}$$

Note that we have substituted $k_c^2 = k_x^2 + k_y^2$ for TE modes. From the above equations and noting that $E_z = 0, H_z \neq 0$ for TE modes, we have,

$$E_x = -j\frac{\omega\mu}{k_x^2 + k_y^2}\frac{\partial H_z}{\partial y} = +\frac{j\omega\mu}{k_x^2 + k_y^2} H_0 k_y \cos k_x x \sin k_y y \sin k_z z$$

$$E_y = \frac{j\omega\mu}{k_x^2 + k_y^2}\frac{\partial H_z}{\partial x} = -\frac{j\omega\mu}{k_x^2 + k_y^2} H_0 k_x \sin k_x x \cos k_y y \sin k_z z$$

$$H_x = -\frac{j\beta}{k_x^2 + k_y^2}\frac{\partial H_z}{\partial x} = \frac{j\beta k_x}{k_x^2 + k_y^2} H_0 \sin k_x x \cos k_y y \sin k_z z$$

$$H_y = -\frac{j\beta}{k_x^2 + k_y^2}\frac{\partial H_z}{\partial y} = \frac{j\beta k_y}{k_x^2 + k_y^2} H_0 \cos k_x x \sin k_y y \sin k_z z$$

Besides, there is no propagation along z-axis and hence β is not a proper term to use. Instead, we should replace it by its more generalized form $-j\beta = \frac{\partial}{\partial z}$. Hence,

$$H_x = \frac{1}{k_c^2}\frac{\partial^2 H_z}{\partial z \partial x} = -\frac{k_x}{k_c^2} H_0 \frac{\partial}{\partial z}\left(\sin k_x x \cos k_y y \sin k_z z\right) = -\frac{k_x k_z}{k_c^2} H_0 \sin k_x x \cos k_y y \cos k_z z$$

$$H_y = \frac{1}{k_c^2}\frac{\partial^2 H_z}{\partial z \partial y} = -\frac{k_y}{k_c^2}\frac{\partial}{\partial z}\left(H_0 \cos k_x x \sin k_y y \sin k_z z\right) = -\frac{k_y k_z}{k_c^2} H_0 \cos k_x x \sin k_y y \cos k_z z$$

where $k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b}, k_z = \frac{l\pi}{d}, k_c^2 = k_x^2 + k_y^2$.

9.5.1 Quality factor for TE_{101} mode:

Quality factor is the most important parameter of a resonator. In the ensuing section, let us calculate Q for TE_{101} mode. The field expressions for TE_{101} mode are

$$E_x = E_z = 0$$

$$E_y = -\frac{j\omega\mu a H_0}{\pi}\sin\frac{\pi}{a}x\sin\frac{\pi}{d}z$$

$$H_x = -\frac{aH_0}{d}\sin\frac{\pi}{a}x\cos\frac{\pi}{d}z$$

$$H_y = 0$$

$$H_z = H_0\cos\frac{\pi}{a}x\sin\frac{\pi}{d}z$$

The quality factor is defined as

$$Q = \omega_0\frac{W_m + W_e}{P_{loss}}$$

At the resonance, it can be shown that

$$W_m = W_e$$

Hence,

$$Q = \omega_0\frac{2W_e}{P_{loss}}$$

Let us calculate electric energy storage inside the rectangular cavity first. Note that fields are time-harmonic functions with sinusoidal variations w.r.t. time and hence its average value over a time period will have a half factor.

$$2W_e = \int_{volume} 2\frac{1}{4}\varepsilon' \left|\vec{E}\right|^2 dv = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d} \frac{1}{2}\varepsilon' \left|\vec{E}\right|^2 dxdydz = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d} \frac{1}{2}\varepsilon' \left|E_y\right|^2 dxdydz$$

$$= \frac{1}{2}\varepsilon' \frac{\omega_0^2\mu^2 a^2 \left|H_0\right|^2}{\pi^2} \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d} \sin^2\frac{\pi}{a}x \sin^2\frac{\pi}{d}z dxdydz$$

$$\because \int_{x=0}^{a} \sin^2\frac{\pi}{a}xdx = \frac{a}{2}, \int_{y=0}^{b} \sin^2\frac{\pi}{b}ydy = \frac{b}{2}, \int_{z=0}^{d} \sin^2\frac{\pi}{d}zdz = \frac{d}{2}$$

$$2W_e = \frac{1}{2}\varepsilon' \frac{\omega_0^2\mu^2 a^2 \left|H_0\right|^2}{\pi^2}\frac{a}{2}b\frac{d}{2} = \frac{\varepsilon'\omega_0^2\mu^2 \left|H_0\right|^2}{\pi^2}\frac{a^3bd}{8}$$

Even though we have assumed perfect metals, generally all metals have some inherent losses due to finite conductivity of the waveguide metal walls. The power lost in the six metallic walls of the rectangular cavity can be calculated as follows.

$$P_{loss} = \int_{walls} \frac{1}{2}R_s \left|\vec{J}_s\right|^2 ds$$

$$\because \hat{n}\times\vec{H} = \vec{J}_s \Rightarrow \left|H_t\right| = \left|\vec{J}_s\right|$$

$$\therefore P_{loss} = \int_{walls} \frac{1}{2}R_s \left|H_t\right|^2 ds$$

$$= \int_{front\ and\ back\ walls} \frac{1}{2}R_s \left|H_t\right|^2 ds + \int_{side\ walls} \frac{1}{2}R_s \left|H_t\right|^2 ds + \int_{top\ and\ bottom\ walls} \frac{1}{2}R_s \left|H_t\right|^2 ds$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b} \frac{1}{2}R_s \left|H_x(z=0)\right|^2 dxdy + 2\int_{y=0}^{b}\int_{z=0}^{d} \frac{1}{2}R_s \left|H_z(x=0)\right|^2 dydz$$

$$+2\int_{x=0}^{a}\int_{z=0}^{d} \frac{1}{2}R_s \left(\left|H_x(y=0)\right|^2 + \left|H_z(y=0)\right|^2\right) dxdz$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} R_s \left|\frac{aH_0}{d}\sin\frac{\pi}{a}x\right|^2 dxdy + \int_{y=0}^{b}\int_{z=0}^{d} R_s \left|H_0 \sin\frac{\pi}{d}z\right|^2 dydz$$

$$+\int_{x=0}^{a}\int_{z=0}^{d} R_s \left(\left|-\frac{aH_0}{d}\sin\frac{\pi}{a}x\cos\frac{\pi}{d}z\right|^2 + \left|H_0\cos\frac{\pi}{a}x\sin\frac{\pi}{d}z\right|^2\right) dxdz$$

$$= R_s \left|\frac{aH_0}{d}\right|^2 \frac{ab}{2} + R_s \left|H_0\right|^2 \frac{bd}{2} + R_s \left|\frac{aH_0}{d}\right|^2 \frac{ad}{4} + R_s \left|H_0\right|^2 \frac{ad}{4}$$

$$= R_s \left|H_0\right|^2 \frac{a^3 b}{2d^2} + R_s \left|H_0\right|^2 \frac{bd}{2} + R_s \left|H_0\right|^2 \frac{a^3}{4d} + R_s \left|H_0\right|^2 \frac{ad}{4}$$

where $R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}}$ is the surface resistivity of metallic walls.

Therefore the quality factor due to conductor losses is

$$Q_c = \omega_0 \frac{\frac{\varepsilon' \omega_0^2 \mu^2 \left|H_0\right|^2}{\pi^2} \frac{a^3 bd}{8}}{R_s \left|H_0\right|^2 \frac{a^3 b}{2d^2} + R_s \left|H_0\right|^2 \frac{bd}{2} + R_s \left|H_0\right|^2 \frac{a^3}{4d} + R_s \left|H_0\right|^2 \frac{ad}{4}}$$

$$= \frac{\varepsilon' \omega_0^3 \mu^2}{\pi^2 R_s} \frac{\frac{a^3 bd}{8}}{\frac{a^3 b}{2d^2} + \frac{bd}{2} + \frac{a^3}{4d} + \frac{ad}{4}} = \frac{\varepsilon' \omega_0^3 \mu^2}{\pi^2 R_s} \frac{a^3 bd}{\frac{4a^3 b}{d^2} + 4bd + \frac{2a^3}{d} + 2ad}$$

$$= \frac{\varepsilon' \omega_0^3 \mu^2}{\pi^2 R_s} \frac{a^3 bd^3}{4a^3 b + 4bd^3 + 2a^3 d + 2ad^3}$$

We could also find the quality factor due to dielectric losses if the cavity is filled with some dielectric. Power loss in the dielectric inside rectangular waveguide cavity can be calculated as

$$P_d = \frac{\omega_0 \varepsilon'' \int_{volume} \left|E\right|^2 dv}{2} = \frac{1}{2} \varepsilon'' \omega_0 \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{d} \left|E_y\right|^2 dxdydz$$

$$= \frac{1}{2} \varepsilon'' \frac{\omega_0^3 \mu^2 a^2 \left|H_0\right|^2}{\pi^2} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{d} \sin^2 \frac{\pi}{a} x \sin^2 \frac{\pi}{d} z dxdydz = \frac{\varepsilon'' \omega_0^3 \mu^2 \left|H_0\right|^2}{\pi^2} \frac{a^3 bd}{8}$$

Therefore the quality factor due to dielectric losses inside the rectangular cavity is given by

$$Q_d = \omega_0 \frac{2W_e}{P_d} = \frac{\dfrac{\varepsilon^{'}\omega_0^3\mu^2 \left|H_0\right|^2}{\pi^2}\dfrac{a^3bd}{8}}{\dfrac{\varepsilon^{"}\omega_0^3\mu^2 \left|H_0\right|^2}{\pi^2}\dfrac{a^3bd}{8}} = \frac{\varepsilon^{'}}{\varepsilon^{"}} = \frac{1}{\tan\delta}$$

Recall that in chapter 2, we have defined the electric permittivity of a material as $\varepsilon = \varepsilon^{'} - j\varepsilon^{"}$ and $\tan\delta = \frac{\varepsilon^{"}}{\varepsilon^{'}}$.

Now that total quality factor can be calculated as

$$Q_{total} = \frac{Q_c Q_d}{Q_c + Q_d}$$

Review question 9.27: How to realize rectangular cavity from rectangular waveguide? What is the additional boundary condition for rectangular cavity from the rectangular waveguide?

Review question 9.28: Write down the expression for dominant mode fields inside a rectangular cavity.

Review question 9.29: Write down the expression of resonant frequency and quality factor for dominant mode of resonance inside a rectangular cavity.

9.6 Circular cavity

Similar analysis could be carried out for the circular cavity of Fig. 9.7 (b). Let us calculate the TE_{mnl} modes ($E_z = 0, H_z \neq 0$). The wave equation in cylindrical coordinate system is given by

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial H_z}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial\phi^2}+\frac{\partial^2 H_z}{\partial z^2}+k^2 H_z=0$$

Applying method of separation of variables, let us assume that $H_z = R(\rho)P(\phi)Z(z)$. Since R, P and Z are functions of only ρ, ϕ and z respectively, then, the above equation reduces to

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho})+\frac{1}{\rho^2 P}\frac{\partial^2 P}{\partial\phi^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+k^2=0$$

Equating the third term in the above equation to a constant $-k_z^2$, we have,

$$\frac{d^2 Z}{dz^2}+k_z^2 Z=0 \Rightarrow Z = A\cos k_z Z + B\sin k_z Z$$

Then the above equation can be reduced to

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho})+\frac{1}{P}\frac{\partial^2 P}{\partial^2\phi^2}+(-k_z^2+k^2)\rho^2=0$$

by multiplying with ρ^2 and substituting $-k_z^2$ instead of $\frac{1}{Z}\frac{d^2 Z}{dz^2}$.

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho})+\frac{1}{P}\frac{\partial^2 P}{\partial^2\phi^2}+k_c^2\rho^2=0$$

In the above equation, we have taken $k^2 - k_z^2 = k_c^2$.

Equating the second term in the above equation to $-k_\phi^2$.

$$\frac{\partial^2 P}{\partial^2\phi^2}+k_\phi^2 P=0 \Rightarrow P = C\cos k_\phi\phi + D\sin k_\phi\phi$$

But for single valued P, we need to choose k_ϕ as an integer n.

$$\Rightarrow P = C\cos n\phi + D\sin n\phi$$

Then,

$$\rho \frac{\partial R}{\partial \rho} + \rho^2 \frac{\partial^2 R}{\partial \rho^2} + ((k_c \rho)^2 - n^2) = 0$$

The solution to the above Bessel's differential equation is given by

$$R = F\, J_n(k_c \rho) + G Y_n(k_c \rho)$$

Note that $Y \to \infty$, $k_c \rho \to \infty$ which is physically not acceptable, choose G=0, then,

$$R = F J_n(k_c \rho)$$

The expression for the longitudinal component of the electric field is

$$H_z(\rho, \phi, z) = RPZ = F J_n(k_c \rho)(C \cos n\phi + D \sin n\phi)\left(A \cos k_z Z + B \sin k_z Z\right)$$

Applying the boundary conditions $H_z = 0$ at $z = 0, d$, we get,

A=0, $\sin k_z d = 0 \Rightarrow k_z d = l\pi$, l is a positive integer

Another boundary condition is that $E_\phi = 0$ at $\rho = a$.

$$\Rightarrow \frac{\partial H_z}{\partial \rho} = 0 \text{ at } \rho = a$$

$$\Rightarrow J_n^{'}(k_c a) = 0 \Rightarrow p_{nm}^{'} = k_c a$$

where $p_{nm}^{'}$ are the mth roots of the first derivative of nth order Bessel's function of first kind.

The expression for the longitudinal component of the electric field is

$$H_z(\rho, \phi, z) = F J_n(k_c \rho)(C \cos n\phi + D \sin n\phi) B \sin k_z z$$

One of the two terms of P can be eliminated. Both have to the same field pattern except for a rotation of 90^0 in the ϕ direction. Also the excitation can be chosen in such a way that only one term can exist at any time. Setting D=0, we can further simplify the above expression as

$$H_z(\rho,\phi,z) = FJ_n(k_c\rho)(C\cos n\phi)B\sin k_z z = H_0(\cos n\phi)J_n(k_c\rho)\sin k_z z$$

We can obtain the other components of electric and magnetic fields from the longitudinal components using the equations listed below.

$$H_\rho = \frac{j\left(\frac{\omega\varepsilon}{\rho}\frac{\partial}{\partial\phi}E_z - \beta\frac{\partial}{\partial\rho}H_z\right)}{k_c^2} \quad (9.16a)$$

$$H_\phi = -\frac{j\left(\omega\varepsilon\frac{\partial}{\partial\rho}E_z + \frac{\beta}{\rho}\frac{\partial}{\partial\phi}H_z\right)}{k_c^2} \quad (9.16b)$$

$$E_\rho = -\frac{j\left(\beta\frac{\partial}{\partial\rho}E_z + \frac{\omega\mu}{\rho}\frac{\partial}{\partial\phi}H_z\right)}{k_c^2} \quad (9.16c)$$

$$E_\phi = \frac{j\left(-\frac{\beta}{\rho}\frac{\partial}{\partial\phi}E_z + \omega\mu\frac{\partial}{\partial\rho}H_z\right)}{k_c^2} \quad (9.16d)$$

Note that $k_c^2 = \left(\frac{p'_{nm}}{a}\right)^2$

From the above equations and noting that $E_z = 0, H_z \neq 0$ for TE modes, we have,

$$H_\rho = -\frac{j\beta\frac{\partial}{\partial\rho}H_z}{k_c^2}$$

$$H_\phi = -\frac{\frac{j\beta}{\rho}\frac{\partial}{\partial\phi}H_z}{k_c^2}$$

$$E_\rho = -\frac{j\frac{\omega\mu}{\rho}\frac{\partial}{\partial\phi}H_z}{k_c^2}$$

$$E_{\phi} = \frac{j\omega\mu \frac{\partial}{\partial \rho} H_z}{k_c^2}$$

Besides, there is no propagation along z-axis and hence β is not a proper term to use. Instead, we should replace it by its more generalized form $-j\beta = \frac{\partial}{\partial z}$. Hence,

$$H_{\rho} = \frac{1}{k_c^2}\frac{\partial^2 H_z}{\partial z \partial \rho} = \frac{H_0 k_z k_c}{k_c^2}(\cos n\phi) J_n^{'}(k_c \rho)\cos k_z z$$

$$H_{\phi} = \frac{1}{\rho k_c^2}\frac{\partial^2 H_z}{\partial z \partial \phi} = \frac{H_0 n k_z}{\rho k_c^2}(\sin n\phi) J_n(k_c \rho)\cos k_z z$$

$$E_{\rho} = -\frac{j\omega\mu}{\rho k_c^2}\frac{\partial}{\partial \phi} H_z = \frac{j\omega\mu H_0 n}{\rho k_c^2}(\sin n\phi) J_n(k_c \rho)\sin k_z z$$

$$E_{\phi} = \frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial \rho} H_z = \frac{j\omega\mu H_0 k_c}{k_c^2}(\cos n\phi) J_n^{'}(k_c \rho)\sin k_z z$$

where $k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b}, k_z = \frac{l\pi}{d}, k_c^2 = k_x^2 + k_y^2$.

Therefore, the resonant frequency for the TE_{nml} modes inside the cylindrical cavity is given by

$$f_r = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{p_{nm}^{'}}{a}\right)^2}$$

Even though TE_{111} is the dominant resonant mode inside the cylindrical cavity, the TE_{011} cylindrical cavity mode is of considerable interest since for an air–filled cavity, the unbounded Q for this mode can be as high as approximately 20,000 to 60,000.

9.6.1 Quality factor for TE_{011} mode:

Quality factor is the most important parameter of a resonator. In the ensuing section, let us calculate Q for TE_{011} mode. The field expressions for TE_{011} mode are

$$H_z(\rho,\phi,z) = H_0 J_0(\frac{p_{01}^{'}}{a}\rho)\sin\frac{\pi}{d}z$$

$$H_\rho = \frac{\pi H_0 a}{p_{01}^{'} d} J_0^{'}(\frac{p_{01}^{'}}{a}\rho)\cos\frac{\pi}{d}z$$

$$H_\phi = 0$$

$$E_\rho = 0 = E_z$$

$$E_\phi = \frac{j\omega\mu H_0 a}{p_{01}^{'}} J_0^{'}(k_c\rho)\sin\frac{\pi}{d}z$$

The quality factor is defined as

$$Q = \omega_0 \frac{W_m + W_e}{P_{loss}}$$

Therefore the quality factor due to dielectric losses inside the circular cavity is the same as that of the rectangular cavity and it is given by

$$Q_d = \omega_0 \frac{2W_e}{P_d} = \frac{\varepsilon^{'}}{\varepsilon^{"}} = \frac{1}{\tan\delta}$$

In order to calculate the quality factor due to losses in the waveguide walls, let us calculate the electric energy storage.

$$2W_e = \int_{volume} 2\frac{1}{4}\varepsilon^{'}\left|\vec{E}\right|^2 dv = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{d}\frac{1}{2}\varepsilon^{'}\left|\vec{E}\right|^2 \rho d\rho d\phi dz = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{d}\frac{1}{2}\varepsilon^{'}\left|E_\phi\right|^2 \rho d\rho d\phi dz$$
$$= \frac{1}{2}\varepsilon^{'}\frac{\omega_0^2\mu^2 a^2\left|H_0\right|^2}{\left(p_{01}^{'}\right)^2}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{d}\left\{J_0^{'}(k_c\rho)\right\}^2\sin^2\frac{\pi}{d}z\rho d\rho d\phi dz$$

$$= \frac{1}{2}\varepsilon^{'}\frac{\omega_0^2\mu^2 a^2\left|H_0\right|^2 \pi d}{\left(p_{01}^{'}\right)^2}\int_{\rho=0}^{a}\left\{J_0^{'}(k_c\rho)\right\}^2 \rho d\rho$$

$$= \frac{1}{2}\varepsilon^{'} \frac{\omega_0^2 \mu^2 a^2 \left|H_0\right|^2 \pi d}{\left(p_{01}^{'}\right)^2 k_c^2} \int\limits_{y=0}^{p_{01}^{'}} \left\{J_0^{'}(y)\right\}^2 y dy$$

$$\because \int\limits_0^{p_{nm}^{'}} \left[\left(J_n^{'}(x)\right)^2 + \frac{n^2}{x^2}\left(J_n(x)\right)^2 \right] x dx = \frac{\left(p_{nm}^{'}\right)^2}{2}\left(1 - \frac{n^2}{\left(p_{nm}^{'}\right)^2}\right) J_n^2\left(p_{nm}^{'}\right)$$

$$\therefore \int\limits_0^{p_{01}^{'}} \left[\left(J_0^{'}(x)\right)^2 \right] x dx = \frac{\left(p_{01}^{'}\right)^2}{2} J_0^2\left(p_{01}^{'}\right)$$

$$2W_e = \frac{1}{2}\varepsilon^{'} \frac{\omega_0^2 \mu^2 a^2 \left|H_0\right|^2 \pi d}{\left(p_{01}^{'}\right)^2 k_c^2} \frac{\left(p_{01}^{'}\right)^2}{2} J_0^2\left(p_{01}^{'}\right) = \frac{\varepsilon^{'} \omega_0^2 \mu^2 a^4 d\pi \left|H_0\right|^2 J_0^2\left(p_{01}^{'}\right)}{4\left(p_{01}^{'}\right)^2}$$

Even though we have assumed perfect metals, generally all metals have some inherent losses due to finite conductivity of the waveguide metal walls. The power lost in the three metallic walls of the circular cavity can be calculated as follows.

$$P_{loss} = \int\limits_{walls} \frac{1}{2} R_s \left|\vec{J}_s\right|^2 ds$$

$$\because \left|H_t\right| = \left|J_s\right|$$

$$\therefore P_{loss} = \int\limits_{walls} \frac{1}{2} R_s \left|H_t\right|^2 ds$$

$$= \int\limits_{front\ and\ back\ walls} \frac{1}{2} R_s \left|H_t\right|^2 ds + \int\limits_{curved\ walls} \frac{1}{2} R_s \left|H_t\right|^2 ds$$

$$= 2\int\limits_{\rho=0}^{a} \int\limits_{\phi=0}^{2\pi} \frac{1}{2} R_s \left|H_\rho(z=0)\right|^2 \rho d\rho d\phi + \int\limits_{\phi=0}^{2\pi} \int\limits_{z=0}^{d} \frac{1}{2} R_s \left|H_z(\rho=a)\right|^2 a d\phi dz$$

$$= 2\int\limits_{\rho=0}^{a} \int\limits_{\phi=0}^{2\pi} \frac{1}{2} R_s \left|\frac{H_0 \pi a}{p_{01}^{'} d} J_0^{'}\left(\frac{p_{01}^{'}}{a}\rho\right)\right|^2 \rho d\rho d\phi + \int\limits_{\phi=0}^{2\pi} \int\limits_{z=0}^{d} \frac{1}{2} R_s \left|H_0 J_0(p_{01}^{'}) \sin\frac{\pi}{d} z\right|^2 a d\phi dz$$

$$= \frac{\pi^2 \left|H_0\right|^2 a^2}{\left(p_{01}^{'}\right)^2 d^2} \int\limits_{\rho=0}^{a} \int\limits_{\phi=0}^{2\pi} R_s \left|J_0^{'}\left(\frac{p_{01}^{'}}{a}\rho\right)\right|^2 \rho d\rho d\phi + \left|H_0\right|^2 \int\limits_{\phi=0}^{2\pi} \int\limits_{z=0}^{d} \frac{1}{2} R_s \left(J_0(p_{01}^{'})\right)^2 \sin^2\frac{\pi}{d} z a d\phi dz$$

$$= \frac{\left|H_0\right|^2 a^2 R_s 2\pi^3}{\left(p_{01}^{'}\right)^2 d^2 k_c^2} \frac{\left(p_{01}^{'}\right)^2}{2} J_0^2\left(p_{01}^{'}\right) + \left|H_0\right|^2 R_s \frac{\pi d a}{2}\left(J_0(p_{01}^{'})\right)^2$$

$$= \frac{\left|H_0\right|^2 R_s 2\pi^3 a^4}{2\left(p_{01}^{'}\right)^2 d^2} J_0^2\left(p_{01}^{'}\right) + \left|H_0\right|^2 R_s \frac{\pi d a}{2}\left(J_0(p_{01}^{'})\right)^2$$

where $R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}}$ is the surface resistivity of metallic walls.

Therefore the quality factor due to conductor losses is

$$Q_c = \omega_0 \frac{\dfrac{\varepsilon^{'}\omega_0^2\mu^2 a^4 d\pi\left|H_0\right|^2 J_0^2\left(p_{01}^{'}\right)}{4\left(p_{01}^{'}\right)^2}}{\dfrac{\left|H_0\right|^2 R_s 2\pi^3 a^4}{2\left(p_{01}^{'}\right)^2 d^2} J_0^2\left(p_{01}^{'}\right) + \left|H_0\right|^2 R_s \dfrac{\pi d a}{2}\left(J_0(p_{01}^{'})\right)^2}$$

$$= \frac{\dfrac{\varepsilon^{'}\omega_0^3\mu^2 a^4 d\pi}{4\left(p_{01}^{'}\right)^2}}{\dfrac{R_s 2\pi^3 a^4}{2\left(p_{01}^{'}\right)^2 d^2} + R_s \dfrac{\pi d a}{2}} = \frac{\varepsilon^{'}\omega_0^3\mu^2 a^4 d^3}{2R_s\left(2\pi^2 a^4 + a\left(p_{01}^{'}\right)^2 d^3\right)}$$

Now that total quality factor can be calculated as

$$Q_{total} = \frac{Q_c Q_d}{Q_c + Q_d}$$

9.7 Rectangular microstrip antenna:

Before, we go into analysis of rectangular microstrip antenna (cavity model), let us try to understand some fundamentals of microstrip antenna (MSA). How does a MSA looks like? MSA is a conducting strip of metal printed on a dielectric substrate, which is situated above a ground plane as depicted in Fig. 9.8(a). The RMSA consists of a ground plane, a dielectric substrate and a rectangular patch. Although RMSA and circular microstrip antenna (CMSA) are most popular MSAs, other shapes such as square, triangular, semicircular, sectoral and annular ring are also used. We will be discussing only RMSA and CMSA in this chapter. The antenna is usually fed by a coaxial cable as shown in Fig. 9.8(b). MSAs have several advantages over other conventional microwave antennas: lightweight and have a small volume and low profile, conformable to various host surfaces, inexpensive and easy to fabricate using printed-circuit technology, compatible with monolithic microwave integrated circuit (MMIC), versatile for polarization (allow both linear and circular polarizations), versatile for resonant frequency (allow for dual, triple and other multiple frequency operations), compact for use in personal mobile communications and so on.

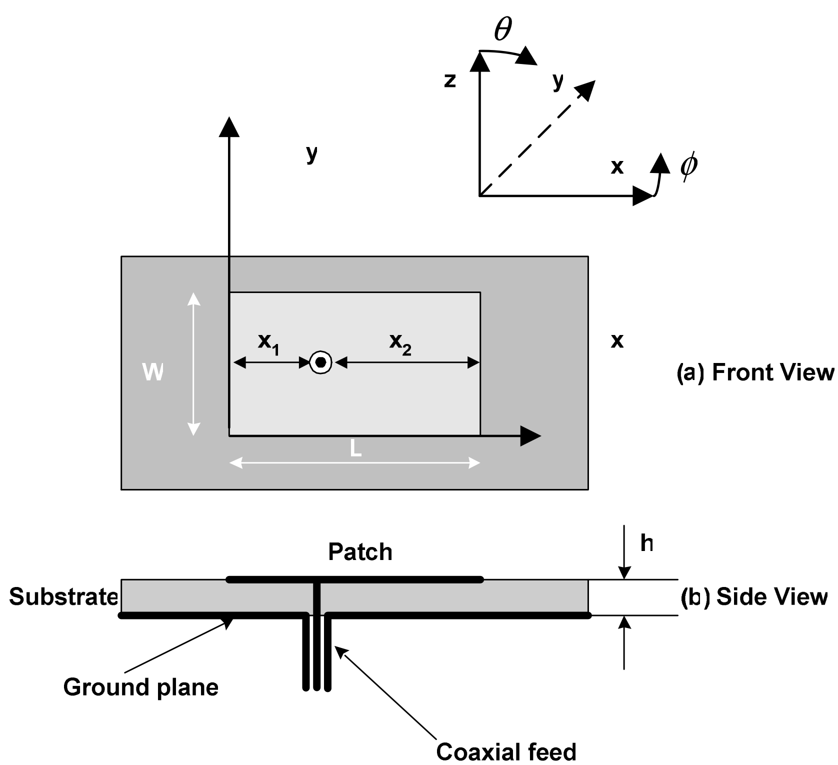


Fig. 9.8 (a) Front view and (b) Side view of rectangular microstrip antenna (RMSA)

MSAs also suffer from some disadvantages compared to other conventional microwave antennas: narrow bandwidth (it is no more a disadvantage for MSA since there are various methods to improve its bandwidth), low gain (MSA arrays can be employed for improving the gain), low power handling capacity, high Q, poor polarization purity, spurious feed radiations, lower efficiency (efficiency is affected by conductor, dielectric and surface wave losses) etc. MSAs are widely used in wireless communications nowadays. Some of the possible applications of MSAs are: (a) telemetry & communication antennas on missiles need to be thin & conformal, MSA can be used (b) antennas for mobile & satellite communications (c) MSA arrays for satellite imaging systems (d) Global System for Mobile Communication (GSM) (e) Global Positioning System (GPS) (f) Communication links between ships & satellites, etc. There are basically two commonly used feeding techniques for MSA: (a) Microstrip feed line: Microstrip feed line is also a conducting strip usually of much smaller width compared to the patch as shown in Fig. 9.9(a). The microstrip feed line is easy to fabricate, simple to match by controlling the inset position and rather simple to model. However as the substrate thickness increases surface waves and spurious feed radiation increases. The width of microstrip line and substrate parameters decides the characteristic impedance of the microstrip line. (b) Coaxial feed line: Coaxial-line feeds, where the inner conductor of the coax is attached to the radiation patch while the outer conductor is connected to the ground plane are also widely used as depicted in Fig. 9.8(b). The coaxial probe feed is also easy to fabricate and match, and it has low spurious radiation. However, it also has narrow bandwidth and it is more difficult to model, especially for thick substrates. The characteristic impedance of the coaxial cable is mainly dependent on the radius of the inner and outer conductor of the coaxial cable as described at the start of this chapter. There are other feeding techniques like electromagnetic coupled feed lines, aperture coupled feed lines, broadband tapered feed lines, etc. which gives broader BW.

Review question 9.30: Explain the structure of MSA.

Review question 9.31: List the advantages and disadvantages of MSA.

Review question 9.32: State some of the applications of MSA.

Review question 9.33: State the two basic feeding methods for MSA and explain them with the help of a diagram.

There are two methods for analysis of RMSA. First is the cavity model. It is applicable for only thin substrates ($h \ll \lambda$) and it can analyze simple structures like RMSA, CMSA, annular-ring (ARMSA), equitriangular (ETMSA) patches. Second and more rigorous method is full-wave analysis based on Method of Moments (MoM)/ Finite Difference Time Domain (FDTD). It is applicable for thicker substrate and other complicated structures like multilayered patches. We will not discuss this method as it is more involved. Instead, we will discuss the cavity model since it gives better understanding of the patch antennas based on its structure and boundary conditions.

9.7.1 Cavity model of RMSA

The volume beneath the patch and the ground plane can be treated as a rectangular cavity loaded with a dielectric with dielectric constant of ε_r (see Fig. 9.9 (b)). The dielectric material of the substrate is assumed to be truncated beyond the edges of patch. This volume looks like the rectangular cavity excited by a coaxial cable. Note some difference between rectangular cavity and the RMSA. In cavity, it was a closed box with six metal walls. In RMSA, the top and bottom walls are metallic walls, whereas, the four side walls are kept open. This gives the chance to the fields inside the cavity to come out of the cavity and hence radiation of fields to outside the RMSA is made possible. If the antenna is excited at a resonant frequency, a strong field is set up inside the cavity and a strong current on the lower surface of the patch which gives rise to a strong radiation. We know that metallic walls are electric walls since the tangential component of the electric field is zero on those walls. What boundary conditions can we expect on the four side walls of the RMSA cavity? Some assumptions:

(a) The field in the interior region do not vary with z since the substrate is very thin (h<<λ) and hence $\frac{\partial}{\partial z} \equiv 0$.

(b) The electric field is z-directed only, and the magnetic field ($\because \vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E} = -\frac{1}{j\omega\mu}\left[\nabla\times\left(\hat{z}E_z\right)\right]$

$= -\frac{1}{j\omega\mu}\left[E_z\left(\nabla\times\hat{z}\right)+\nabla E_z\times\hat{z}\right] = -\frac{1}{j\omega\mu}\nabla E_z\times\hat{z} = \frac{1}{j\omega\mu}\hat{z}\times\nabla E_z$ from Maxwell's curl equation and example 1.3) has only the transverse components in the region bounded by the patch and ground plane. That's why we have electric walls at the top and bottom (tangential components of the electric field are zero).

Note the relation between the magnetic field and the surface current density on the lower surface of the patch ($\vec{J}_s = -\hat{z}\times\vec{H}$). The electric current in the patch boundary perimeter has no component normal to the edge of the patch ($\hat{n}\bullet\vec{J}_s = 0 \Rightarrow \hat{n}\bullet\vec{J}_s = \hat{n}\bullet\left(-\hat{z}\times\vec{H}\right) = -\vec{H}\bullet\left(\hat{n}\times\hat{z}\right) = H_t = 0$), which means the tangential component of the magnetic field is negligible along the side walls since the magnetic field is approximately independent of z inside the cavity (only normal component of the magnetic fields exist on the four side walls) or magnetic walls (walls in which tangential components of the magnetic field are zero).

The field regions may be divided into two: interior and exterior. Hence, the interior region of the patch is modeled as a cavity bounded by electric walls at the top and bottom, and a magnetic wall all along the four side walls. From our knowledge in EM theory (refer to chapter 8), the magnetic vector potential should satisfy the wave equation in the interior region of the cavity.

$$\nabla^2\vec{A}+k^2\vec{A}=0$$

For the TM_z mode fields (since magnetic field has only x- and y- components inside the cavity, magnetic vector potential should have only z component), we have,

$$\vec{A}=A_z\hat{z}$$

$$\Rightarrow \nabla^2 A_z + k^2 A_z = 0$$

Applying the method of separation of variables, A_z takes the form,

$$A_z=(A_1\cos k_x x + B_1 \sin k_x x)(A_2 \cos k_y y + B_2 \sin k_y y)(A_3 \cos k_z z + B_3 \sin k_z z)$$

Note that

$$\vec{B}=\mu\vec{H}=\nabla\times\vec{A}$$

$$\Rightarrow H_x\hat{x}+H_y\hat{y}+H_z\hat{z}=\frac{1}{\mu}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y}\\ 0 & 0 & A_z\end{vmatrix}$$

Hence,

$$H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} = \frac{k_y}{\mu}(A_1\cos k_x x + B_1\sin k_x x)(-A_2\sin k_y y + B_2\cos k_y y)(A_3\cos k_z z + B_3\sin k_z z)$$

$$H_y = -\frac{1}{\mu}\frac{\partial A_Z}{\partial x} = -\frac{k_x}{\mu}(-A_1\sin k_x x + B_1\cos k_x x)(A_2\cos k_y y + B_2\sin k_y y)(A_3\cos k_z z + B_3\sin k_z z)$$

$$H_z = 0$$

Also $\nabla\times\vec{H} = j\omega\vec{E}$

$$\Rightarrow\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ H_x & H_y & H_z\end{vmatrix} = j\omega\varepsilon(E_x\hat{x} + E_y\hat{y} + E_z\hat{z})$$

$$\Rightarrow -\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \Rightarrow E_x = -\frac{1}{j\omega\varepsilon}\frac{\partial H_y}{\partial z}$$

$$= \frac{k_x k_z}{j\omega\varepsilon\mu}(-A_1\sin k_x x + B_1\cos k_x x)(A_2\cos k_y y + B_2\sin k_y y)(-A_3\sin k_z z + B_3\cos k_z z)$$

$$\Rightarrow -\frac{\partial H_x}{\partial z} = j\omega\varepsilon E_y \Rightarrow E_y = -\frac{1}{j\omega\varepsilon}\frac{\partial H_x}{\partial z}$$

$$= -\frac{1}{j\omega\varepsilon}\frac{k_y k_z}{\mu}(A_1\cos k_x x + B_1\sin k_x x)(-A_2\sin k_y y + B_2\cos k_y y)(A_3\sin k_z z + B_3\cos k_z z)$$

$$\Rightarrow \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \Rightarrow E_z = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$= \frac{1}{j\omega\varepsilon}\begin{pmatrix}\frac{k_x^2}{\mu}(A_1\cos k_x x + B_1\sin k_x x)(A_2\cos k_y y + B_2\sin k_y y)(A_3\cos k_z z + B_3\sin k_z z) - \\ \frac{k_y^2}{\mu}(A_1\cos k_x x + B_1\sin k_x x)(A_2\cos k_y y + B_2\sin k_y y)(A_3\cos k_z z + B_3\sin k_z z)\end{pmatrix}$$

$$= \frac{1}{j\omega\varepsilon}\left(\frac{k_x^2 - k_y^2}{\mu}(A_1\cos k_x x + B_1\sin k_x x)(A_2\cos k_y y + B_2\sin k_y y)(A_3\cos k_z z + B_3\sin k_z z)\right)$$

Applying boundary conditions,

$E_x = 0\big|_{z=0,h}$ (ground plane and metal patch are electric walls, note that E_y is also the tangential component of the electric field in these two walls, we can also apply the boundary condition for this as well and it will give the same result), $H_y = 0\big|_{x=0,L}$ (front and back walls are magnetic walls, H_z is also the tangential component of the magnetic field, but it is zero for TM fields), $H_x = 0\big|_{y=0,W}$ (side walls are magnetic walls, H_z is also the tangential component of the magnetic field, but it is zero for TM fields)

$$E_x = 0\big|_{z=0,h} \Rightarrow B_3 = 0 \text{ and } k_z = \frac{p\pi}{h}$$

$$H_y = 0\big|_{x=0,L} \Rightarrow B_1 = 0 \text{ and } k_x = \frac{m\pi}{L}$$

$$H_x = 0\big|_{y=0,W} \Rightarrow B_2 = 0 \text{ and } k_y = \frac{n\pi}{W}$$

where $m = n = p \neq 0$ and $m, n, p = 0, 1, 2, ...$

Now we can write

$$A_z = A_{mnp} \cos k_x x \cos k_y y \cos k_z z$$

Therefore, the field expressions may be simplified as follows:

$$H_x = -\frac{k_y}{\mu} A_{mnp} (\cos k_x x)(\sin k_y y)(\cos k_z z)$$

$$H_y = \frac{k_x}{\mu} A_{mnp} (\sin k_x x)(\cos k_y y)(\cos k_z z)$$

$$H_z = 0$$

$$E_x = \frac{k_x k_z}{j\omega\varepsilon\mu} A_{mnp} (\sin k_x x)(\cos k_y y)(\sin k_z z)$$

$$E_y = \frac{1}{j\omega\varepsilon} \frac{k_y k_z}{\mu} A_{mnp} (\cos k_x x)(\sin k_y y)(\sin k_z z)$$

$$E_z = \frac{1}{j\omega\varepsilon} \frac{k_x^2 - k_y^2}{\mu} A_{mnp} (\cos k_x x)(\cos k_y y)(\cos k_z z)$$

$$\because k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu\varepsilon$$

We can find the resonant frequency for TM_{mnp} modes inside the rectangular cavity as follows.

$$\Rightarrow f_r\Big|_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{L}\right)^2+\left(\frac{n\pi}{W}\right)^2+\left(\frac{p\pi}{h}\right)^2}$$

Dominant mode for L>W>h is TM_{100} mode and has the resonant frequency $(f_r)_{100} = \frac{1}{2L\sqrt{\mu\varepsilon}}$.

For this dominant mode, the field expressions are

$$H_x = 0$$

$$H_y = \frac{\pi}{\mu L} A_{100} (\sin\frac{\pi}{L}x)$$

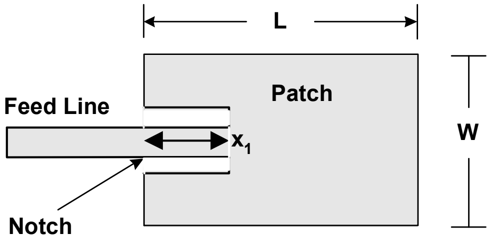
$$H_z = 0$$

$$E_x = 0$$

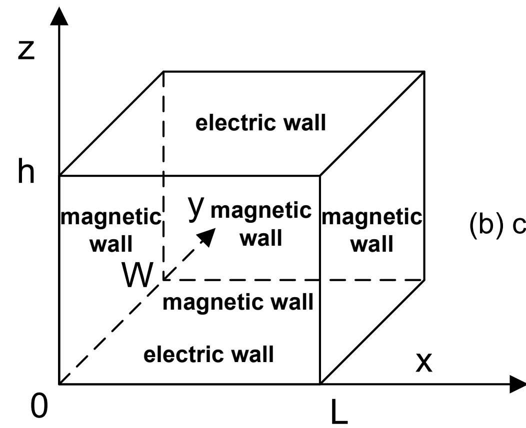
$$E_y = 0$$

$$E_z = \frac{1}{j\omega\varepsilon}\frac{\pi^2}{\mu L^2} A_{100} (\cos\frac{\pi}{L}x)$$

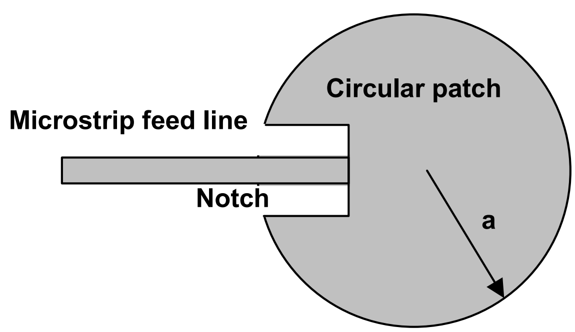
In addition L>W>L/2>h, the next higher mode is $(f_r)_{010} = \frac{1}{2W\sqrt{\mu\varepsilon}}$. If however L>L/2>W>h the second dominant mode is the $TM^z{}_{200}$ mode whose resonant frequency is given by $(f_r)_{200} = \frac{1}{L\sqrt{\mu\varepsilon}}$.



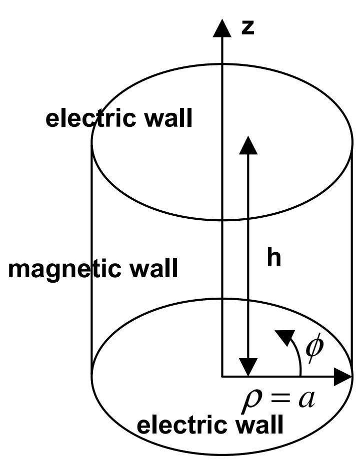
(a) Microstrip line feed



(b) cavity model of RMSA



(c) Top view of CMSA



(d) Cavity model of CMSA

Fig. 9.9 (a) Microstrip fed RMSA (b) Cavity model of RMSA (c) Top view of CMSA (d) Cavity model of CMSA

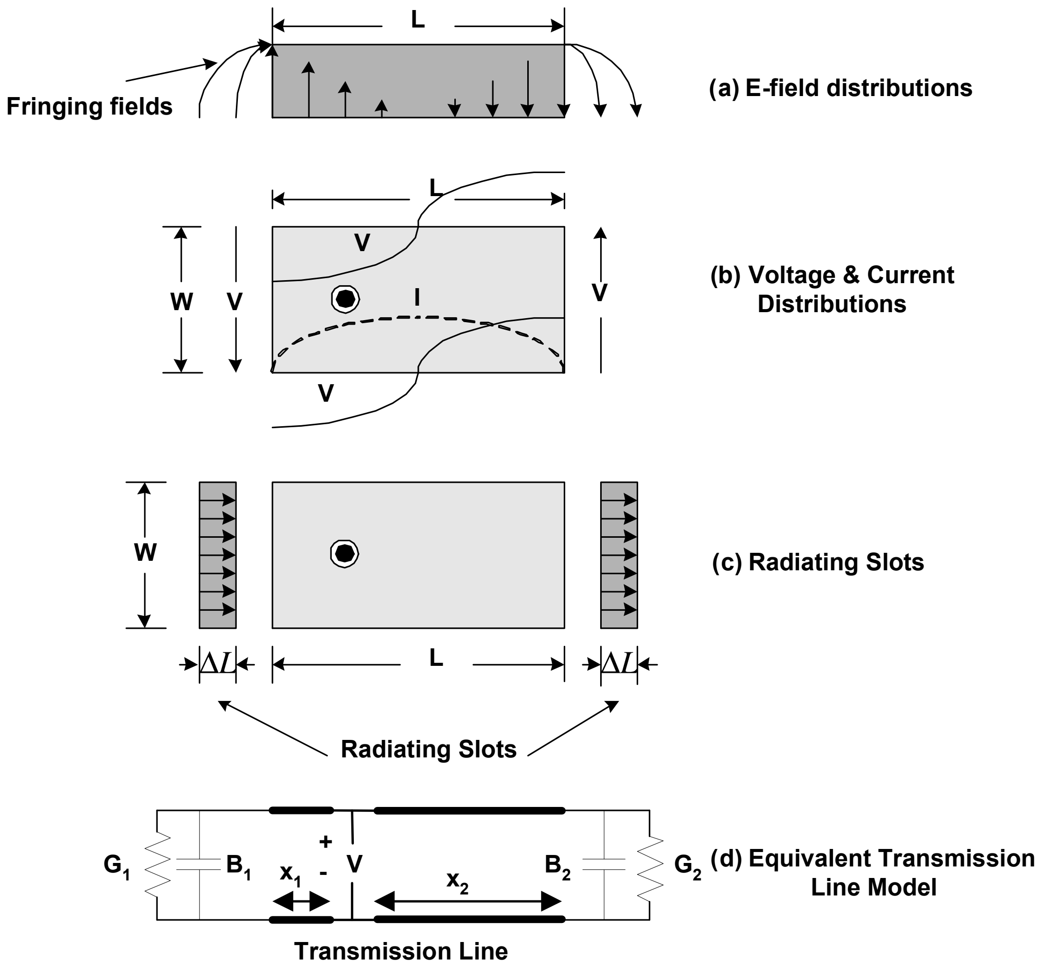


Fig. 9.10 (a) E-field distributions (b) Voltage and Current distributions (c) Radiating Slots and (d) Equivalent transmission line model for the dominant mode of RMSA

9.7.2 Design of RMSA

(a) Design formulae

- For an RMSA to be an efficient radiator, W should be taken equal to a half wavelength corresponding to the average of the two dielectric mediums (i.e., substrate and air).

$$W = \frac{c}{2f_0\sqrt{\frac{\varepsilon_r + 1}{2}}}$$

where f_0 is the resonant frequency of the MSA, ε_r is the relative dielectric constant of the substrate.

W is usually chosen larger than L for higher bandwidth, but it should be less than 2L.

- The value of ε_e is slightly less than ε_r, because the fringing fields around the periphery of the patch are not confined in the dielectric substrate but are also spread in the air as shown in Fig. 9. 10 (a).

$$\varepsilon_e = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2\sqrt{1 + \frac{12h}{W}}}$$

- Due to the fringing fields at the two edges, the effective length of the RMSA in Fig. 9.10 (c) is given by

$$L_e = L + 2\Delta L = \frac{c}{2f_0\sqrt{\varepsilon_e}}$$

- Because of the fringing effects, electrically the patch of the MSA looks greater than its physical dimension (extension of ΔL on both sides). A very practical approximate relation for the normalized extension of the length is given by

$$\frac{\Delta L}{h} = 0.412 \frac{(\varepsilon_{eff} + 0.3)(\frac{w}{h} + 0.264)}{(\varepsilon_{eff} - 0.258)(\frac{w}{h} + 0.8)}$$

where h is the thickness of the substrate and it is assumed to be much smaller than the dimensions of the antenna.

For the two types of feeding techniques, we need to locate the position x where the impedance of the antenna is equal to 50 Ohms since most of the coaxial cables has an impedance of 50 Ohms. For microstrip feed lines of MSA, we need to choose the width of the microstrip lines (W_m) accordingly.

$$\frac{W_m}{h} = \begin{cases} \frac{8e^A}{e^{2A} - 2}; \frac{W_m}{h} < 2 \\ \frac{2}{\pi}\left[B - 1 - \ln(2B - 1) + \frac{\varepsilon_r - 1}{2\varepsilon_r}\left\{\ln(B - 1) + 0.39 - \frac{0.61}{\varepsilon_r}\right\}\right]; \frac{W_m}{h} > 2 \end{cases}$$

$$A = \frac{Z_0}{60}\sqrt{\frac{\varepsilon_r + 1}{2}} + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\left(0.23 + \frac{0.11}{\varepsilon_r}\right);$$

$$B = \frac{377\pi}{2Z_0\sqrt{\varepsilon_r}}$$

In the above equation, Z_0 is usually chosen as 50 Ohms, h and ε_r are fixed for a particular substrate.

After calculation of the dimensions of the patch, the design process is continued with the matching of the antenna resistance to 50 Ohm of the input line. For impedance matching with the microstrip feed line, inset feeding technique is generally used. Position of the inset feed point is calculated as follows:

- Antenna impedance and feed location: Along the width of the patch, voltage is maximum and current is minimum due to the open end as shown in Fig. 9.10 (b). Hence the value of resistance of the antenna is maximum at the edges and zero at the center of the patch. Coaxial feed is placed in the middle of the width to avoid excitation of the orthogonal higher order TM_{010} mode and cross-polarization. For a given mode, it can be shown that the resonant input impedance (R_{in}) of a RMSA is directly proportional to the square of the cavity mode electric field at the feed point. For the dominant TM_{100} mode of the RMSA, $E_z = \frac{1}{j\omega\varepsilon}\frac{\pi^2}{\mu L^2}A_{mnp}(\cos\frac{\pi}{L}x)$.

For coaxial feed at a distance x_1 from the edge towards the center as shown in Fig. 9.8 (a), the input impedance of the RMSA for the dominant TM_{100} mode at resonance for can be approximately calculated as:

$$R_{in} = R_e \cos^2(\frac{\pi x_1}{L}) \text{ for } 0 \le x_1 \le L/2$$

where $R_e = \frac{1}{2(G_r \pm G_m)}$, + sign is usually chosen due to the odd field distribution between the radiating slots for the dominant TM_{100} mode.

G_m is the mutual conductance which accounts for mutual coupling between the two slots and is given by

$$G_m = \frac{1}{120\pi^2}\int_0^\pi \left[\frac{\sin\left(\frac{k_0 W\cos\theta}{2}\right)}{\cos\theta}\right]^2 J_0(k_0 L\sin\theta)\sin^3\theta d\theta$$

A formula for calculation of G_r from Balanis,

$$G_r = \frac{I_1}{120\pi^2}$$

$$I_1 = \int_0^\pi \left[\frac{\sin\left(\frac{k_0 W\cos\theta}{2}\right)}{\cos\theta}\right]^2 \sin^3\theta d\theta = -2+\cos(X)+XS_i(X)+\frac{\sin(X)}{X}; X = k_0 W$$

But it has been observed experimentally that 4^{th} square law gives a better and accurate position of the feed point. Hence,

$$R_{in} = R_e \cos^4(\frac{\pi x_1}{L}) \text{ for } 0 \leq x_1 \leq L/2$$

We will use the 4^{th} square law in our design of RMSA. In the above formula, Re is the resonant resistance of the patch antenna (at the edge) and R_{in} is the required input resistance, which is 50 Ohm for this case. Note that the antenna resistance (Rin) is highest at the edge and lowest at the center. The inset feed point x_1 along the length is a point in between the edge and center of the patch antenna. The feed point location is chosen at the mid-point of the width of the patch antenna to avoid higher order mode excitations and for polarization purity.

Design steps for RMSA:

- Given ε_r, h in cm, f_r in Hz
- Determine L and W of the patch
- Find W, ε_{eff}, ΔL, L_e and $L=L_e-2\Delta L$
- Find the width of the microstrip feed line
- Find the feed location

Review question 9.34: Write down the steps for designing RMSA.

<u>*Example 9.4*</u>

Write a MATLAB program to design RMSA.

It should prompt for the following inputs.

Enter the resonant frequency in GHz:

Enter the dielectric constant of the substrate:

Enter the thickness of the substrate:

Enter the characteristic impedance needed for the microstrip transmission line:

It should give the following outputs:

Ereff =

Width_microstrip_line =

W_Patch =

L_Patch =

Inset =

Solution:

```
% The following program computes some important parameters of a rectangular
% microstrip  patch antenna
%The inputs to the program are the thickness of the substrate h , the
%desired resonant frequency fr, the characteristic impedance of the microstrip feedline,
%the dielectric constant of the substrate Er.The layout is shown below.It uses inset feed
%
%
%                    <----------     L     ------------->
%
%                    ____________________
%         |       |                        ____|    |    Feed  microstrip
%         |       |                       |______|____________
%         |       |                                                   |
%         W       |                                W0                 |
%         |       |                         ____________________|
```

```
%            |          |                              |______
%            |          |____________________|
%                                                        <------>
%                                                            y0
clear all;
% warning off MATLAB:divideByZero;
fr1=input('Enter the resonant frequency in GHz : ');
Er=input('\n Enter the dielectric constant of the substrate : ');
h1=input('\n Enter the thickness of the substrate in mm : ')
Zc=input('\n Enter the characteristic impedance needed for the microstrip transmission
line: ')
fr=fr1*1e9;
h=h1*1e-3;
% Velocity of light in vacuum
c0=299792458;
%Wavelength in free space Lambda
Lambda0=c0/fr;
% Free space wave impedance
Z0=120*pi;
%Determining the width of the microstrip transmission line needed
   A=(pi*sqrt(2*(Er+1)))*(Zc/Z0)+((Er-1)/(Er+1))*(0.23+0.11/Er);
   ratio1=4/(0.5*exp(A)-exp(-A));
   B=(pi/(2*sqrt(Er)))*(Z0/Zc);
```

```
    ratio2=((Er-1)/(pi*Er))*(log(B-1)+0.39-0.61/Er)+(2/pi)*(B-1-log(2*B-1));

    if ratio1 <= 2

        ratio = ratio1;

    else ratio = ratio2;

    end

    W_tln=h*ratio;

% Width of the patch

W=(c0/(2*fr))*sqrt(2/(Er+1));

% The effective dielectric constant

Ereff=(Er+1)/2+((Er-1)/2)*(power((1+12*h/W),-0.5));

% Wavelength in effective medium

Lambda=Lambda0/sqrt(Ereff);

% Speed of light in effective medium

c=c0/sqrt(Ereff);

%Length of microstrip TLN for a 360 degree phase shift

L_tln=Lambda

% The correction of length delL due to the fringing

delL=0.412*h*((Ereff+0.3)*(W/h+0.264))/((Ereff-0.258)*(W/h+0.8));

% Correct length needed

L=Lambda/2-2*delL;

%The feed  inset required

% Calculation of the conductance

 k0=2*pi/Lambda0;
```

```
 X=k0*W;
%Calculation of the integral of sin(x)/x
 syms t;
 f=sin(t)./t;
 SiX=quadl(inline('f'),eps,X);
 I1=-2+cos(X)+sin(X)./X+X*SiX;
 G1= I1/(120*(pi^2));
Rin0=1/(2*G1);
% Use cos^4 dependence
y0=(L/pi)*acos((power((Zc/Rin0),0.25)));
disp('All the values are in microns');
Ereff
Width_microstrip_line=W_tln/1e-6
W_Patch=W/1e-6
L_Patch=L/1e-6
Inset=y0/1e-6
```

Example 9.5

Show that for an RMSA, $R_{in} = R_e \cos^2\left(\beta x_1\right)$ for the dominant mode of operation where R_{in} is the input resistance of the antenna at the feed point x_1 and R_e is the edge resistance of the RMSA.

Solution:

Using the Transmission line model of RMSA (Fig. 9.10 (d)) and assuming that the two slots are identical $\left(G_1 + jB_1 = G_2 + jB_2 = G + jB\right)$, we have,

$$Y_{in} = Y_0 \frac{G + jB + jY_0 \tan(\beta x_1)}{Y_0 + j(G + jB)\tan(\beta x_1)} + Y_0 \frac{G + jB + jY_0 \tan(\beta(L - x_1))}{Y_0 + j(G + jB)\tan(\beta(L - x_1))}$$

Also note that the length of the RMSA is approximately half the guided wavelength $(\beta L = \pi)$ at the resonant frequency, hence,

$$Y_{in} = Y_0 \frac{G + jB + jY_0 \tan(\beta x_1)}{Y_0 + j(G + jB)\tan(\beta x_1)} + Y_0 \frac{G + jB - jY_0 \tan(\beta x_1)}{Y_0 - j(G + jB)\tan(\beta x_1)}$$

$$= \frac{2(G + jB)Y_0^2 + 2Y_0^2 (G + jB)\tan^2(\beta x_1)}{Y_0^2 + (G + jB)^2 \tan^2(\beta x_1)}$$

$$= \frac{2(G + jB) + 2(G + jB)\tan^2(\beta x_1)}{1 + \frac{(G + jB)^2}{Y_0^2} \tan^2(\beta x_1)}$$

Since the microstrip patch is a microstrip line of large width, it will have a very low characteristic impedance and hence, a high characteristic admittance.

$$Y_{in} \cong 2(G + jB) + 2(G + jB)\tan^2(\beta x_1) = \frac{2(G + jB)}{\cos^2(\beta x_1)}$$

Now let us find the edge resistance (x_1=0).

$$Y_e = Y_0 \frac{G + jB}{Y_0} + Y_0 \frac{G + jB}{Y_0} = 2(G + jB)$$

Therefore,

$$R_{in} = R_e \cos^2(\beta x_1)$$

9.7 Circular Microstrip Antenna

9.7.1 Design of CMSA

(a) Introduction

The next most popular MSA is CMSA. It can be used as a single element or in arrays. The modes supported by the CMSA can be found by treating the patch, ground plane and the substrate between the two as a circular cavity. The cavity is composed of two perfect electric conductors in the top and bottom to represent the patch and the ground plane and by a cylindrical perfect magnetic conductor around the circular periphery of the cavity (to model an open circuit). For the CMSA there is only one degree of freedom to control (radius of the patch). Changing the radius doesn't change the order of modes however it changes the resonant frequency of the patch.

(b) Resonant frequencies

The resonant frequencies of the CMSA can be obtained using the formula:

$$f_r = \frac{X'_{nm} c}{2\pi a\sqrt{\varepsilon_r}}$$

where X'_{nm} is the mth root of the derivative of the Bessel function of order n. For the dominant TM_{11} the resonant frequency is given by

$$f_r = \frac{1.8412c}{2\pi a\sqrt{\varepsilon_r}}$$

The resonant frequency doesn't take into account fringing. Fringing makes the patch look electrically larger and hence the effective radius a_e should replace the a in the previous equation. The relation between the a and a_e is as follows:

$$a_e = a\left\{1+\frac{2h}{\pi a\varepsilon_r}\left[\ln\left(\frac{\pi a}{2h}\right)+1.7726\right]\right\}^{1/2} \Rightarrow a = \frac{a_e}{\left\{1+\frac{2h}{\pi a\varepsilon_r}\left[\ln\left(\frac{\pi a}{2h}\right)+1.7726\right]\right\}^{1/2}}$$

Therefore the resonant frequency for the dominant mode should be expressed as

$$f_r = \frac{1.8412c}{2\pi a_e\sqrt{\varepsilon_r}} \Rightarrow a_e = \frac{8.791\times10^9}{f_r\sqrt{\varepsilon_r}}cm$$

(c) Procedures for design of CMSA

- Given ε_r, h in cm, f_r in Hz
- Determine actual radius a of the patch, use the following relation:

$$a = \frac{a_e}{\left\{1+\frac{2h}{\pi a\varepsilon_r}\left[\ln\left(\frac{\pi a}{2h}\right)+1.7726\right]\right\}^{1/2}};a_e = \frac{8.791\times10^9}{f_r\sqrt{\varepsilon_r}}$$

Remember that h must be in cm. In the first iteration, choose a in the denominator of the above equation equal to a_e, then in the other iterations use the value a from the previous iteration and in the numerator is always the value of a_e. Continue the iteration till you get a value of a, which is convergent.

(d) Feed location

The fields of the TM_{11} mode produce a virtual short circuit at the center of the patch. Experience shows that the 50 Ohm feed line is located from the center at about one-third of the radius. The radial line along which the feed is located determines the direction of the linear polarization. Derneryd gives an approximate expression for the radial impedance variation:

$$R_{in} = R_e\frac{J_1^2(k_\varepsilon\rho)}{J_1^2(k_\varepsilon a)}$$

where R_e is the edge resistance, ρ is the radial distance and J_1 is the Bessel function of the first kind, $k_\varepsilon = k\sqrt{\varepsilon_r}$. The edge resistance for dominant mode m=1 can be calculated using the following equation:

$$R_e = \frac{1}{G_t} = \frac{1}{G_{rad} + G_c + G_d}$$

where

$$G_{rad} = \frac{(k_0 a_e)^2}{480}\int_0^{\pi/2}\left[B_P^2(k_0 a\sin\theta) + \cos^2\theta B_M^2(k_0 a\sin\theta)\right]\sin\theta d\theta;$$

$$B_P(x) = J_{m-1}(x) + J_{m+1}(x); B_M(x) = J_{m-1}(x) - J_{m+1}(x)$$

$$G_c = \frac{\pi(\pi\mu_0 f_r)^{-3/2}}{4h^2\sqrt{\sigma}}\left[(ka_e)^2 - m^2\right]$$

$$G_d = \frac{\tan\delta}{4\mu_0 h f_r}\left[(ka_e)^2 - m^2\right]$$

As usual the contribution from the first term is the major, other two terms may be neglected in our analysis.

Review Question 9.35: Write down the steps for designing CMSA.

9.7.2 Cavity Model of CMSA

The modes supported by the circular patch antenna can be found by treating the region between the circular patch and the ground plane as a circular cavity as in the case of RMSA (see Fig. 9.9 (d)). The modes that are supported by a CMSA, whose substrate height is small, is TM^z where 'z' is taken perpendicular to the patch. The cavity is composed of two perfect electric conductors at the top and bottom to represent the patch and the ground plane and by a cylindrical perfect magnetic conductor around the cylindrical side walls of the cavity. The dielectric material of the substrate is assumed to be truncated beyond the extent of the patch. The vector wave equation to be solved is simplified to $\nabla^2 A_z + k^2 A_z = 0$ subjected to boundary conditions $E_\rho \big|_{z=0,h} = 0$ and $H_\phi \big|_{\rho=a} = 0$. Applying the method of separation of variables like in the case of cylindrical cavity, we can get the solution of the wave equation as follows:

$$A_z = (A_1 J_n(k_c \rho) + B_1 Y_n(k_c \rho))(A_2 \cos k_\phi \phi + B_2 \sin k_\phi \phi)(A_3 \cos k_z z + B_3 \sin k_z z)$$

Since, $Y \to \infty$ as $k_c \rho \to \infty$ which is physically not acceptable, so, we should choose B_1=0.

$$A_z = (A_1 J_n(k_c \rho))(A_2 \cos k_\phi \phi + B_2 \sin k_\phi \phi)(A_3 \cos k_z z + B_3 \sin k_z z)$$

We also have the standard relation between the magnetic field intensity, magnetic flux density and magnetic vector potential as

$$\vec{B} = \mu \vec{H} = \nabla \times \vec{A}$$

$$\Rightarrow H_\rho \hat{\rho} + H_\phi \hat{\phi} + H_z \hat{z} = \frac{1}{\mu\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

For this case, A_ρ =0 and ρA_ϕ =0, hence,

$$H_\rho = \frac{1}{\rho\mu} \frac{\partial A_z}{\partial \phi},$$

$$H_\phi = -\frac{1}{\mu} \frac{\partial A_z}{\partial \rho},$$

$$H_z = 0$$

Also from the Maxwell curl equation, we have,

$$\nabla \times \vec{H} = j\omega\varepsilon \vec{E}$$

$$\Rightarrow \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & H_\phi & 0 \end{vmatrix} = j\omega\varepsilon(E_\rho\hat{\rho} + E_\phi\hat{\phi} + E_z\hat{z})$$

Equating the three vector components of the above equation, we can get the expressions of electric field components in terms of magnetic field components which could be further expressed in terms of magnetic vector potential A_z as follows.

$$\Rightarrow -\frac{1}{\rho}\frac{\partial H_\phi}{\partial z} = j\omega\varepsilon E_\rho \Rightarrow E_\rho = \frac{1}{j\omega\varepsilon\mu\rho}\frac{\partial^2 A_z}{\partial z \partial \rho};\frac{\partial H_\phi}{\partial z} = j\omega\varepsilon E_\phi \Rightarrow E_\phi = \frac{1}{j\omega\varepsilon\mu\rho}\frac{\partial^2 A_z}{\partial z \partial \phi};$$

$$\frac{1}{\rho}(\frac{\partial H_\phi}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}) = j\omega\varepsilon E_z \Rightarrow E_z = -\frac{1}{\rho^2 j\omega\varepsilon\mu}(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \phi^2})A_z$$

Now applying the two boundary conditions, we get,

(a) $E_\rho = 0\big|_{z=0,h} \Rightarrow A_3 = 0,\ k_z = \frac{l\pi}{h}$

(b) $H_\phi = 0\big|_{\rho=a},\ \Rightarrow J_n'(k_c\rho) = 0,\ k_c a = p_{mn}'$

Some typical values of the roots of the differentiation of the n[th] order Bessel's functions of the first kind are $p_{11}' = 1.8412$, $p_{21}' = 3.054$, $p_{01}' = 3.8318$ and $p_{31}' = 4.2012$.

Note that k_ϕ is an integer i.e., $k_\phi = n$ in order to have single valued P. Hence, using the above two conditions, we can further simplify the expression for A_z as follows.

$$A_z = A_1 J_n(k_c\rho)(A_2\cos n\phi + B_2 \sin n\phi)A_3 \cos k_z z$$

Neglecting the second term of $(A_2\cos n\phi + B_2 \sin n\phi)$, we have,

$$A_z = A_{mnp} J_n(k_c\rho)(\cos n\phi)\cos k_z z$$

In the above equation, we have assumed that the multiplication of three arbitrary constants $(A_1 \times A_2 \times A_3)$ is equal to a new arbitrary constant (A_{mnp}).

Also we have, $k_c^2 + k_z^2 = k^2 = \omega_r^2 \mu\varepsilon$. From this relation, we can find the resonant frequency for various TM_{mn0} modes inside the CMSA as follows:

$$(f_r)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}(\frac{p'_{mn}}{a})$$

A particular case of the above expression is for TM_{mn0} mode inside CMSA is $(f_r)_{110} = \frac{1.8412}{2\pi a\sqrt{\mu\varepsilon}}$.

We may also calculate the quality factor of RMSA and CMSA due to conductor and dielectric losses. The process to obtain Q is quite similar to that of rectangular and circular cavity. This is left as an exercise for the readers (see Exercise 9.9 and 9.10). Besides, we could also obtain the far field radiation fields from the cavity fields. This is out of scope of this book. The readers may refer to any of the book on microstrip antenna mentioned in the references.

9.8 Summary

In this chapter, we have studied three disparate topics viz. waveguide, cavity and micrsotrip antenna. But we have observed that the analysis of rectangular waveguide, rectangular cavity and rectangular micrsotrip antenna is quite similar (only the boundary conditions change). Similarly, the analysis of circular waveguide, circular cavity and circular micrsotrip antenna is almost the same (only the boundary conditions change). In fact, the analysis of rectangular waveguide and circular waveguide, rectangular cavity and circular cavity and rectangular microstrip antenna and circular microstrip antenna is basically same except for the change in the coordinate system from the Cartesian coordinate system to Cylindrical coordinate system. Hence, if we know how to analyze, any one of the above devices/structures, we can always do the analysis of the remaining devices/structures in a similar way. That is the idea behind clubbing together three disparate topics under one chapter.

Exercises

Exercise 9.1

How to obtain vector wave equations from the two Maxwell's curl equations?

Exercise 9.2

What is the wave equation inside rectangular waveguide?

Exercise 9.3

What is the wave equation inside circular waveguide?

Exercise 9.4

How to obtain other transversal components of $\vec{E}$ and $\vec{H}$ fields (say E_x, E_y, H_x, H_y) from the longitudinal components (say E_z, H_z)?

Exercise 9.5

How to obtain the transversal components $E_\rho, E_\phi, H_\rho, H_\phi$ in cylindrical coordinates when longitudinal components of electric and magnetic fields E_z, H_z are given?

Exercise 9.6

Find the first three propagating modes of a hollow circular waveguide of radius 0.5 cm.

Exercise 9.7

Design a RMSA using FR4 substrate with dielectric constant of 4.4 and h=1.6mm so as to resonate at 2.4 GHz for WLAN applications. Find

(a) the input impedance

(b) the position of the inset feed-point where the input impedance is 50 Ohms

(c) the width of 50 Ohms microstrip feed line

Exercise 9.8

Design a CMSA using FR4 substrate with dielectric constant of 4.4 and h=1.6mm so as to resonate at 900 MHz for GSM mobile communications. Find

(a) the input impedance

(b) the position of the feed-point where the input impedance is 50 Ohms

Exercise 9.9

Find the quality factor due to conductor and dielectric loss of a RMSA for the dominant mode of operation.

Exercise 9.10

Find the quality factor due to conductor and dielectric loss of a CMSA for the dominant mode of operation.